Solving Gauge Invariant Systems without Gauge Fixing: the Physical Projector in 0+1 Dimensional Theories

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Abstract

The projector onto gauge invariant physical states was recently constructed for arbitrary constrained systems. This approach, which does not require gauge fixing nor any additional degrees of freedom beyond the original ones—two characteristic features of all other available methods for quantising constrained dynamics—is put to work in the context of a general class of quantum mechanical gauge invariant systems. The cases of SO(2) and SO(3) gauge groups are considered specifically, and a comprehensive understanding of the corresponding physical spectra is achieved in a straightforward manner, using only standard methods of coherent states and group theory which are directly amenable to generalisation to other Lie algebras. Results extend by far the few examples available in the literature from much more subtle and delicate analyses implying gauge fixing and the characterization of modular space.

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1 Introduction

There is no need to emphasize the importance of the gauge invariance concept in physics and mathematics today, both in the classical as well as in the quantum realms. Ever since Dirac's classification of constraints in terms of first- and second-class ones[1], the former associated to the generators of local gauge symmetries, quite a number of quantisation methods have been developed for such systems[2], culminating with the so-called BFV-BRST Hamiltonian or Lagrangian approaches[5], both in the operatorial and in the path-integral frameworks.

Nevertheless, in spite of all the mathematical elegance and profound insight offered by these methods, it may seem somewhat odd that in order to circumvent the unwanted effects of unphysical gauge variant negative norm quantum states present in a manifestly spacetime covariant dynamics, which is what the BFV-BRST approaches are tailored to achieve, unavoidably quite a heavy machinery of ghosts and ghosts for ghosts has to be introduced generically. This is in sharp contrast with Dirac's own approach to covariant quantisation, in which physical gauge invariant states are simply those which are annihilated by the quantum gauge generators, without the need to introduce any further degrees of freedom beyond the original ones.

An additional issue usually not properly addressed within the BFV-BRST approaches is that of the required gauge fixing of the gauge freedom, with the ensuing possible Gribov problems[6], either of a local or of a global character (or both) on the space of gauge orbits of the system[1]. In fact, whatever the gauge fixing procedure, this issue of possible Gribov problems must certainly be considered, if only to conclude that it is absent, which is typically not the case. Indeed, the BFV-BRST approaches cannot guarantee the absence of such Gribov ambiguities given a specific gauge fixing procedure, while a correct description of the physics of non perturbative effects can be achieved certainly only for a gauge fixing free of any Gribov ambiguity.

Recently[7], the projector for physical gauge invariant states was constructed for general constrained systems[2], within the coherent state approach to canonical quantisation[10]. This construction proceeds within Dirac's original covariant quantisation of such systems, without the need either of additional degrees of freedom, nor of any gauge fixing procedure. Subsequently, the physical projector approach was shown[11] to be generally free of any Gribov ambiguities, and to lead naturally to the correct integration over the space of gauge orbits of the system.

To explore further the advantages of the simplicity afforded through the physical projection operator and coherent states, it is worthwhile to show how that approach also allows for an efficient resolution of some gauge invariant quantised dynamics. The systems considered in the present paper are matter coupled Yang-Mills theories in 0+1 dimensions, thus corresponding to quantum mechanical systems rather than quantum field theories. Nevertheless, their dynamics remains both simple and rich enough to allow for explicit solutions which still demonstrate the advantages of the physical projector approach over more established ones. In a certain sense, these models may also be viewed as deriving from the dimensional reduction to time dependent configurations only, of pure Yang-Mills theories in higher dimensions. In this respect, it is intriguing to remark also that similar compactifications have recently become of much interest in the context of dualities and the non perturbative dynamics of M-theory[12], thus obviously offering a whole new field to which the present methods could be applied, including fermionic degrees of freedom.

More specifically, systems with SO(2) and SO(3) local gauge invariance will be considered presently. In the first instance, both the physical spectrum as well as the wave functions for all

[1] For a detailed discussion, see for example pp. 143-153 and Chapter 4 of Ref.[3].

[2] Such a physical projector had already been introduced[8] in the case of some simple gauge invariant systems[9], similar to those analysed in the present paper.

gauge invariant states will be constructed explicitly. In the SO(3) case, only the physical spectrum is derived, leaving further issues to be explored in the general context of this class of models for any simple compact Lie algebra. Quite accidently, some of the models considered here have recently been analysed in a more traditional setting, addressing both the issues of the necessary gauge fixing and the ensuing possible Gribov problems very carefully and properly[13]. The physical spectra and wave functions determined in that work enable a direct comparison with our results, and where they may be compared, they do indeed agree. However, our approach is simpler and more straightforward, in that it applies well known techniques of coherent states, integration over group manifolds and representation theory, while that of Ref.[13] must dissect with great care the structure of the space of gauge orbits of the system and the explicit resolution of Schrödinger's equation on that space which possesses typical conical singularities.

The paper is organised as follows. First, an SO(2) gauge invariant model is considered, by providing its general motivations (Sect.2), considering its classical dynamics (Sect.3), its canonical quantisation (Sect.4), solving its spectrum of gauge invariant physical states (Sect.5), accounting for the reasons of the degeneracies of that spectrum (Sect.6), and finally constructing the generating function of the wave functions of its physical states (Sect.7). Similar considerations are then partly developed in the case of a model with non abelian SO(3) gauge symmetry, introduced in Sect.8 and whose physical spectrum is determined in Sect.9, but leaving the detailed investigation of the wave functions of these states for later work. Finally, some conclusions and prospects for further developments are presented in Sect.10, while specific results of use in the body of the paper are provided in two Appendices.

2 The SO(2) Gauge Invariant Model

The degrees of freedom of the model are a real variable $\lambda(t)$ and a collection of $2d$ real variables $q^a_i(t)$ $(a=1,2; i=1,2,\cdots,d)$, whose dynamics is determined from the Lagrangian function[3]

$$L=\frac{1}{2g^2}\left[\dot{q}^a_i-\lambda\epsilon^{ab}q^b_i\right]^2-V(q^a_i)\quad , \tag{1}$$

with

$$V(q^a_i)=\frac{1}{2}\omega^2\,q^a_iq^a_i\quad . \tag{2}$$

Here, g and ω are arbitrary parameters with appropriate physical dimensions, and ϵ^{ab} is the two dimensional antisymmetric tensor such that $\epsilon^{12}=+1$.

Clearly, the model is that of a collection of d real "scalar fields" q^a_i in 0+1 dimensions, in the defining two dimensional representation of SO(2), or rather such a collection of d non relativistic particles propagating in a two dimensional space with a mass normalised to unity. These particles are coupled to the single time component λ of the SO(2) gauge field, while their coordinates have been rescaled by the gauge coupling constant g. The kinetic term is the square of the associated gauge covariant time derivative, while $V(q^a_i)$ could be any SO(2) invariant potential of the scalar fields, namely invariant under rotations in the index $a=1,2$. Quite obviously, the generator of SO(2) gauge transformations is the total two dimensional angular momentum of the system of d particles, which must thus vanish identically at all times for gauge invariant configurations.

Since the gauge field λ in 0+1 dimensions may be gauged away, these d particles interact with one another through the potential $V(q^a_i)$. The choice $V(q^a_i)=0$ would correspond to those

[3]The summation convention over repeated indices is implicit throughout, including in squared quantities.

particles being free. However, we choose to work rather with the harmonic potential given in (2), since the system then not only becomes readily tractable as a collection of d spherical harmonic oscillators, but also provides for a natural regularisation of its mass spectrum which then gains also a mass gap as in any *bona fide* confining gauge theory. Indeed, the dimensional reduction to 0+1 dimensions of a pure *non abelian* Yang-Mills theory leads to a Lagrangian generalising that in (1), with (the generalisation of) λ then corresponding to the time component of the gauge field and q_i^a to its space components, while the potential $V(q_i^a)$ is then *quartic* in these components in the case of a non abelian gauge group[4]. Since these non abelian interactions are presumed to be precisely those responsible for a finite mass gap and confinement, introducing a quadratic potential in the abelian SO(2) case above does go some way in modeling the non abelian case. Likewise for dimensionally reduced non abelian models, introducing a quadratic gauge invariant potential as in (2) turns the reduced model into a collection of anharmonic oscillators with an angular frequency ω as a free regularisation parameter, while nevertheless, the quartic term may be removed altogether without spoiling the non abelian gauge symmetry of the reduced system.

The type of model defined by (1) has been considered previously in the literature[8, 9, 11], and quite recently again in Ref.[13]. In particular, the latter work addressed specifically the SO(2) and SO(3) gauge invariant models for $d = 2$ particles, and solved their gauge invariant physical spectra having also chosen the harmonic potential (2). Consequently in the case $d = 2$, our results are directly comparable, even though the methods are completely different. Indeed, Ref.[13] carefully determines the modular space of gauge orbits of the system, and quantises it by solving the Schrödinger equation on that space, which thus possesses conical singularities. In contradistinction, ours is a method which relies on well established techniques of coherent states and group theory, without having to address the subtle issues of gauge fixing and possible Gribov ambiguities. Exploring further the advantages of the present approach is certainly a worthwhile programme, including higher dimensional gauge invariant theories.

3 The SO(2) Model: Classical Analysis

The SO(2) gauge transformations leaving the Lagrangian (1) invariant, are given by

$$\lambda'(t) = \lambda(t) + \dot\theta(t) \quad , \quad q'^a_i(t) = U^{ab}\left(\theta(t)\right) q^b_i(t) \quad , \tag{3}$$

where $\theta(t)$ is an arbitrary time dependent function[5], while $U^{ab}\left(\theta(t)\right)$ is the rotation matrix

$$\left[U^{ab}\left(\theta(t)\right)\right] = \left(\begin{array}{cc} \cos\theta(t) & \sin\theta(t) \\ -\sin\theta(t) & \cos\theta(t) \end{array}\right) \quad . \tag{4}$$

Indeed, under (3), the covariant time derivatives $(\dot q^a_i - \lambda\epsilon^{ab}q^b_i)$ transform in the same manner as do the variables q^a_i.

The Euler-Lagrange equations of motion for the particle positions q^a_i are simply

$$\frac{d}{dt}\left[\dot q^a_i - \lambda\epsilon^{ab}q^b_i\right] - \lambda\epsilon^{ab}\left[\dot q^b_i - \lambda\epsilon^{bc}q^c_i\right] = -g^2\omega^2 q^a_i \quad , \quad a = 1,2;\ i = 1,2,\cdots,d \quad , \tag{5}$$

while that for the gauge degree of freedom λ is the constraint

$$\epsilon^{ab}q^a_i\left[\dot q^b_i - \lambda\epsilon^{bc}q^c_i\right] = 0 \quad , \tag{6}$$

[4]Note that dimensional reduction of a 1+1 dimensional theory does not lead to such a quartic potential[11].

[5]The variations (3) do not induce a surface term in the Lagrangian (1).

corresponding to an identically vanishing total angular momentum of the system of d particles.

Clearly, given the gauge transformations (3), any configuration for $\lambda(t)$ may be gauged away through a transformation with parameter $\theta(t)$ such that

$$\theta(t)=\theta_0-\int_{t_0}^{t}dt'\,\lambda(t')\quad , \tag{7}$$

where θ_0 is the value of $\theta(t)$ at $t=t_0$, an arbitrary integration constant. Consequently, in the gauge $\lambda(t)=0$, the equations of motion reduce to those of a collection of d harmonic oscillators in two dimensions,

$$\ddot{q}^a_i=-g^2\omega^2 q^a_i\quad , \tag{8}$$

constrained to have a vanishing total angular momentum, $\epsilon^{ab}q^a_i\dot{q}^b_i=0$.

The Hamiltonian analysis of constraints proceeds as in the usual Yang-Mills case[2]. The "fundamental first-order Hamiltonian description"[6] is then provided by the first-order Lagrangian

$$L=\dot{q}^a_i p^a_i-\frac{1}{2}g^2 p^a_i p^a_i-\frac{1}{2}\omega^2 q^a_i q^a_i-\lambda\epsilon^{ab}p^a_i q^b_i=\dot{q}^a_i p^a_i-H_0-\lambda\phi\quad , \tag{9}$$

where p^a_i are the momenta conjugate to the coordinates q^a_i, whose Poisson brackets with the latter are canonical, while λ turns out to be the Lagrange multiplier for the first-class constraint ϕ generating the local SO(2) gauge invariance of the system,

$$\phi=\epsilon^{ab}p^a_i q^b_i\quad , \tag{10}$$

which commutes with the first-class Hamiltonian $H_0=\left[g^2(p^a_i)^2+\omega^2(q^a_i)^2\right]/2$, namely $\{H_0,\phi\}=0$.

Infinitesimal Hamiltonian gauge transformations generated by ϕ are thus defined by

$$\delta_\eta q^a_i=\{q^a_i,\eta\phi\}=\eta\epsilon^{ab}q^b_i\quad,\quad \delta_\eta p^a_i=\{p^a_i,\eta\phi\}=\eta\epsilon^{ab}p^b_i\quad,\quad \delta_\eta\lambda=\dot{\eta}\quad , \tag{11}$$

where $\eta(t)$ is an arbitrary (infinitesimal) function. Clearly, these transformations exponentiate to finite SO(2) gauge transformations acting on the Hamiltonian variables as follows,

$$q'^a_i(t)=U^{ab}\left(\theta(t)\right)q^b_i(t)\quad,\quad p'^a_i(t)=U^{ab}\left(\theta(t)\right)p^b_i(t)\quad,\quad \lambda'(t)=\lambda(t)+\dot{\theta}(t)\quad , \tag{12}$$

where $\theta(t)$ is an arbitrary rotation angle, and the matrix $U^{ab}\left(\theta(t)\right)$ is given in (4). These transformations thus coincide with the SO(2) gauge transformations in the Lagrangian formulation of the system.

The Hamiltonian equations of motion are simply[7]

$$\dot{q}^a_i=g^2p^a_i+\lambda\epsilon^{ab}q^b_i\quad,\quad \dot{p}^a_i=-\omega^2q^a_i+\lambda\epsilon^{ab}p^b_i\quad , \tag{13}$$

together with the gauge constraint $\phi=\epsilon^{ab}p^a_i q^b_i=0$. Here again, the local SO(2) gauge freedom enables one to gauge away the Lagrange multiplier λ, thereby leading in the $\lambda=0$ gauge to the usual Hamiltonian equations of motion for harmonic oscillators but constrained to have a vanishing total angular momentum $\phi=0$.

[6] This notion is introduced on pp. 124-134 of Ref.[3].

[7] Note how the terms linear in λ in these expressions show how the Lagrange multiplier is indeed related to a SO(2) rotation added onto the straightforward time evolution generated by the time derivative.

4 The SO(2) Model: Canonical Quantisation

Canonical quantisation of the model is straightforward, through the canonical commutation relations,

$$(\hat{q}_i^a)^\dagger = \hat{q}_i^a \quad , \quad (\hat{p}_i^a)^\dagger = \hat{p}_i^a \quad , \quad \left[\hat{q}_i^a, \hat{p}_j^b\right] = i\hbar\delta^{ab}\delta_{ij} \quad , \tag{14}$$

and the quantum Hamiltonian and gauge generator,

$$\hat{H} = \hat{H}_0 + \lambda(t)\hat{\phi} \quad , \quad \hat{H}_0 = \frac{1}{2}g^2\hat{p}_i^a\hat{p}_i^a + \frac{1}{2}\omega^2\hat{q}_i^a\hat{q}_i^a \quad , \quad \hat{\phi} = \epsilon^{ab}\hat{p}_i^a\hat{q}_i^b \quad , \tag{15}$$

which obviously commute also at the quantum level, $[\hat{H}_0, \hat{\phi}] = 0$, thus establishing gauge invariance of the quantised system as well.

As usual (see Appendix A), it is convenient to introduce creation and annihilation operators in these cartesian coordinates, defined by

$$\alpha_i^a = \sqrt{\frac{\omega}{2\hbar g}}\left[\hat{q}_i^a + i\frac{g}{\omega}\hat{p}_i^a\right] \quad , \quad \alpha_i^{a\dagger} = \sqrt{\frac{\omega}{2\hbar g}}\left[\hat{q}_i^a - i\frac{g}{\omega}\hat{p}_i^a\right] \quad , \tag{16}$$

such that

$$\left[\alpha_i^a, \alpha_j^{b\dagger}\right] = \delta^{ab}\delta_{ij} \quad , \tag{17}$$

while in terms of normal ordered expressions, and the excitation level operator $\hat{N} = \alpha_i^{a\dagger}\alpha_i^a$,

$$\hat{H}_0 = \hbar g\omega\left[\alpha_i^{a\dagger}\alpha_i^a + \frac{1}{2}2d\right] = \hbar g\omega\left[\hat{N} + d\right] \quad , \quad \hat{\phi} = i\hbar\epsilon^{ab}\alpha_i^{a\dagger}\alpha_i^b \quad . \tag{18}$$

Related to these definitions as well as to the phase space degrees of freedom q_i^a and p_i^a, the following complex variables may also be considered,

$$z_i^a = \sqrt{\frac{\omega}{2\hbar g}}\left[q_i^a + i\frac{g}{\omega}p_i^a\right] \quad , \quad \bar{z}_i^a = \sqrt{\frac{\omega}{2\hbar g}}\left[q_i^a - i\frac{g}{\omega}p_i^a\right] \quad . \tag{19}$$

Given this cartesian basis of degrees of freedom, it is possible to introduce generating vectors spanning the whole representation space of the quantised system, namely Fock states as well as holomorphic or phase space coherent states (see Appendix A). However, as discussed in Appendix A, in the case of a two dimensional harmonic oscillator with circular symmetry, it proves efficient to rather work in a helicity-like basis, taking thus advantage of the corresponding SO(2)=U(1) symmetry, which in the present instance coincides also with the local gauge invariance of the system.

In terms of the cartesian quantities introduced above, the helicity annihilation and creation operators are thus defined by

$$\alpha_i^\pm = \frac{1}{\sqrt{2}}\left[\alpha_i^1 \mp i\alpha_i^2\right] \quad , \quad \alpha_i^{\pm\dagger} = \frac{1}{\sqrt{2}}\left[\alpha_i^{1\dagger} \pm i\alpha_i^{2\dagger}\right] \quad , \tag{20}$$

such that

$$\left[\alpha_i^+, \alpha_j^{+\dagger}\right] = \delta_{ij} = \left[\alpha_i^-, \alpha_j^{-\dagger}\right] \quad , \tag{21}$$

while one also has

$$z_i^\pm = \frac{1}{\sqrt{2}}\left[z_i^1 \mp iz_i^2\right] \quad , \quad \bar{z}_i^\pm = \frac{1}{\sqrt{2}}\left[\bar{z}_i^1 \pm i\bar{z}_i^2\right] \quad . \tag{22}$$

Consequently, the following relations hold,

$$z_i^a \alpha_i^{a\dagger} = z_i^+ \alpha_i^{+\dagger} + z_i^- \alpha_i^{-\dagger} \quad , \quad |z|^2 \equiv |z_i^a|^2 = |z_i^+|^2 + |z_i^-|^2 \quad , \tag{23}$$

as well as,

$$\hat{H}_0 = \hbar g\omega \left[\alpha_i^{+\dagger}\alpha_i^+ + \alpha_i^{-\dagger}\alpha_i^- + d\right] = \hbar g\omega \left[\hat{N} + d\right] \quad , \quad \hat{\phi} = -\hbar\left[\alpha_i^{+\dagger}\alpha_i^+ - \alpha_i^{-\dagger}\alpha_i^-\right] \quad . \tag{24}$$

Correspondingly, the helicity Fock state basis is given by

$$|n_i^\pm> = \prod_i \frac{1}{\sqrt{n_i^+!\, n_i^-!}} \left(\alpha_i^{+\dagger}\right)^{n_i^+} \left(\alpha_i^{-\dagger}\right)^{n_i^-} |0> \quad , \tag{25}$$

$|0>$ being the usual normalised vacuum state, while the holomorphic helicity coherent states (strictly, holomorphic up to a factor) are defined by

$$|z_i^\pm> = e^{-\frac{1}{2}|z|^2}\, e^{z_i^+\alpha_i^{+\dagger}}\, e^{z_i^-\alpha_i^{-\dagger}}\, |0> \quad . \tag{26}$$

In particular, the Fock states $|n_i^\pm>$ diagonalise both the Hamiltonian $\hat{H}_0$ and the gauge generator $\hat{\phi}$, since,

$$\hat{H}_0|n_i^\pm> = \hbar g\omega\left[\sum_i (n_i^+ + n_i^-) + d\right]|n_i^\pm> \quad , \quad \hat{\phi}|n_i^\pm> = -\hbar\sum_i (n_i^+ - n_i^-)\,|n_i^\pm> \quad . \tag{27}$$

Hence, the physical energy spectrum of gauge invariant states, namely those states annihilated by the operator $\hat{\phi}$, is given by

$$E_n = \hbar g\omega\,(2n+d) \quad , \quad n = 0,1,2,\ldots \quad , \tag{28}$$

and at energy level E_n, is spanned by those states $|n_i^\pm>$ which satisfy the left- and right-handed helicity matching condition,

$$\sum_i n_i^+ = n = \sum_i n_i^- \quad . \tag{29}$$

Nevertheless, this conclusion is a far cry from a complete characterization of the physical spectrum of the model, since it does not provide a rationale for the degeneracies in the energy spectrum when $d \geq 2$, nor does it give any handle on the explicit contruction of these states. As will be shown in the next three Sections, the physical projection operator[7] provides precisely the adequate tool to address these issues, still entirely within Dirac's quantisation of the model which does not entail any gauge fixing.

As a matter of fact, the advantages of the physical projector approach are best exploited by working with the coherent state representation, rather than the Fock space one. As a first illustration, let us consider the action of the SO(2) gauge generator $\hat{\phi}$ on the helicity coherent states $|z_i^\pm>$. In the same way as is demonstrated in Appendix A, it is clear that for finite SO(2) transformations, one simply finds

$$e^{-\frac{i}{\hbar}\theta\hat{\phi}}\,|z_i^\pm> = e^{i\theta[\alpha_i^{+\dagger}\alpha_i^+ - \alpha_i^{-\dagger}\alpha_i^-]}\,|z_i^\pm> = |e^{\pm i\theta} z_i^\pm> \quad , \tag{30}$$

or equivalently, in terms of the cartesian variables,

$$e^{-\frac{i}{\hbar}\theta\hat{\phi}}\,|z_i^a> = |U^{ab}(\theta) z_i^b> \quad , \tag{31}$$

$U^{ab}(\theta)$ being the SO(2) rotation matrix given in (4). Since the physical projection operator involves precisely a gauge invariant integration of all such finite transformations over the entire gauge group, these simple properties of coherent states clearly indicate that they provide for an ideal calculational tool in this context.

5 The SO(2) Model: the Gauge Invariant Partition Function

The physical projection operator[7] is a quantum object which, being applied onto any quantity, constructs a gauge invariant one simply by averaging over the manifold of the gauge symmetry group all finite gauge transformations generated by the first-class constraints of a system. Quite clearly, this averaging procedure projects out of any quantity its gauge variant components, while leaving unaffected its gauge invariant and thus physically observable ones. In particular, acting on any given state in the representation space of the quantised system, the physical projector constructs a gauge invariant state annihilated by the gauge constraints, namely a physical state, hence its name.

In the case of the SO(2) model, the group manifold is simply the unit circle parametrised by the rotation angle $0 \leq \theta \leq 2\pi$ introduced in (4) and associated to rotations in the index $a = 1, 2$. Consequently, since the normalised invariant integration measure over the SO(2) group reduces to $\int_0^{2\pi} d\theta/2\pi$, for this model the physical projector is given by the following operator,

$$\mathbb{E} = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{-\frac{i}{\hbar}\theta\hat{\phi}} \quad , \tag{32}$$

such that,

$$\mathbb{E}^\dagger = \mathbb{E} \quad , \quad \mathbb{E}^2 = \mathbb{E} \quad . \tag{33}$$

Obviously, the physical projector $\mathbb{E}$ commutes with any gauge invariant operator, such as the first-class Hamiltonian $\hat{H}_0$, and leaves invariant any physical state annihilated by the gauge generator $\hat{\phi}$.

One direct application of the physical projector is in the description of the time evolution of physical states. Given a specific Lagrange multiplier function $\lambda(t)$, the time evolution operator of the quantised system is given by[8],

$$U(t_2, t_1) = e^{-\frac{i}{\hbar}\int_{t_1}^{t_2} dt[\hat{H}_0 + \lambda(t)\hat{\phi}]} \quad . \tag{34}$$

By definition, this evolution propagates in time gauge variant as well as gauge invariant states. It is in order to consistently propagate gauge invariant states only that all other methods of quantisation have been developed, whether the so-called reduced phase space ones or the BFV-BRST extended ones[2]. With the help of the physical projector however, a unitary consistent time propagation of physical states only is readily achieved by projecting out any gauge variant contribution, namely,

$$U_{\rm phys}(t_2, t_1) = U(t_2, t_1)\mathbb{E} = \mathbb{E}\, U(t_2, t_1)\mathbb{E} \quad . \tag{35}$$

Moreover, due to the integration over the angle θ in (32), any contribution stemming from the Lagrange multiplier $\lambda(t)$ is averaged out as well, leaving the physical evolution operator built from the first-class Hamiltonian $\hat{H}_0$ only, and thus finally

$$U_{\rm phys}(t_2, t_1) = e^{-\frac{i}{\hbar}(t_2 - t_1)\hat{H}_0}\, \mathbb{E} = \mathbb{E}\, e^{-\frac{i}{\hbar}(t_2 - t_1)\hat{H}_0}\, \mathbb{E} \quad . \tag{36}$$

This form makes it clear that the spectral decomposition of the physical evolution operator $U_{\rm phys}(t_2, t_1)$ reads as follows,

$$U_{\rm phys}(t_2, t_1) = \sum_{E_n, \mu_n} e^{-\frac{i}{\hbar}(t_2 - t_1)E_n}\, |E_n, \mu_n >< E_n, \mu_n| \quad , \tag{37}$$

[8]Since $\hat{H}_0$ and $\hat{\phi}$ have a vanishing commutator, the time-ordered exponential indeed reduces to this expression.

where $|E_n, \mu_n>$ are the orthonormalised physical states of energy $E_n = \hbar g\omega(2n + d)$, μ_n being a multi-index labelling the associated energy degeneracies. In view of the energy spectrum of physical states, this same expression also reads as

$$U_{\rm phys}(t_2, t_1) = e^{-i(t_2-t_1)g\omega d} \sum_{n,\mu_n} e^{-i(t_2-t_1)g\omega(2n)} \, |E_n, \mu_n >< E_n, \mu_n| \quad . \tag{38}$$

Consequently, given an arbitrary complex parameter[9] x, the basic operator which encompasses all the properties of gauge invariant physical states is[10]

$$\sum_{n,\mu_n} x^{2n} \, |E_n, \mu_n >< E_n, \mu_n| \; = \; \mathbb{E} \, x^{\hat{N}} \, \mathbb{E} \; = \; x^{\hat{N}} \, \mathbb{E} \quad , \tag{39}$$

$\hat{N}$ being the total excitation level operator introduced in (18) and (24). Indeed, the trace of that operator is the partition function for physical states, since, based on the above identification and for $|x| < 1$, we have

$$\mathrm{Tr}\, x^{\hat{N}} \, \mathbb{E} = \sum_{n=0}^{\infty} \, d_n \, x^{2n} \quad , \tag{40}$$

where the coefficients d_n ($n = 0, 1, 2 \ldots$) are the degeneracies of physical states at excitation level $\hat{N} = 2n$, thus at the energy level $E_n = \hbar g\omega(2n + d)$.

Similarly, even only diagonal matrix elements of the operator (39) provide a generating function for the wave functions of the gauge invariant states. For example in the helicity coherent state basis, one has

$$< z_i^{\pm} | x^{\hat{N}} \, \mathbb{E} | z_i^{\pm} > = \sum_{n,\mu_n} \, x^{2n} \, | < z_i^{\pm} | E_n, \mu_n > |^2 \quad . \tag{41}$$

Hence, up to a constant phase, which is always a matter of convention anyway, wave functions of physical states may be determined from the diagonal matrix elements of the operator (39).

In conclusion, all that remains to be done in order to solve the physical spectrum of the system and its time evolution, is the evaluation of the operator (39), for which the coherent state representation is most convenient for reasons already pointed out in the previous Section.

In particular, let us consider the determination of the partition function. Since

$$\mathrm{Tr}\, x^{\hat{N}} \, \mathbb{E} = \int_0^{2\pi} \frac{d\theta}{2\pi} \int \prod_{\pm,i} \frac{dz_i^{\pm} d\bar{z}_i^{\pm}}{\pi} < z_i^{\pm} | x^{\hat{N}} e^{i\theta[\alpha_i^{+\dagger}\alpha_i^{+} - \alpha_i^{-\dagger}\alpha_i^{-}]} | z_i^{\pm} > \quad , \tag{42}$$

one simply has

$$\mathrm{Tr}\, x^{\hat{N}} \, \mathbb{E} = \int_0^{2\pi} \frac{d\theta}{2\pi} \int \prod_{\pm,i} \frac{dz_i^{\pm} d\bar{z}_i^{\pm}}{\pi} < z_i^{\pm} | x e^{\pm i\theta} z_i^{\pm} > \, e^{-\frac{1}{2}(1-|x|^2)|z|^2} \quad . \tag{43}$$

The gaussian integrals stemming from the coherent state matrix elements are readily performed (see Appendix A), leading to

$$\mathrm{Tr}\, x^{\hat{N}} \, \mathbb{E} = \int_0^{2\pi} \frac{d\theta}{2\pi} \, \frac{1}{[1 - xe^{i\theta}]^d \, [1 - xe^{-i\theta}]^d} \quad , \tag{44}$$

[9]Possibly such that $|x| < 1$ in order to ensure uniform convergence of the quantity to be introduced presently.

[10]While the physical evolution operator $U_{\rm phys}(t_2, t_1)$ thus corresponds to $x^{\hat{N}+d} \mathbb{E}$ evaluated for $x = e^{-i(t_2-t_1)g\omega}$.

and thus, finally,

$$\mathrm{Tr}\, x^{\hat{N}} \mathbb{E} = \sum_{n=0}^{\infty} d_n\, x^{2n} \quad , \tag{45}$$

with the degeneracies given by

$$d_n = \left[\frac{(d-1+n)!}{(d-1)!\, n!} \right]^2 \quad , \quad n = 0, 1, 2, \dots \quad . \tag{46}$$

In the case $d = 1$, these degeneracies $d_n = 1$ are trivial at all physical excitations levels $n = 0, 1, 2, \ldots$, a conclusion which is indeed obvious given the left- and right-helicity matching condition (29) on the physical spectrum.

When $d = 2$, the degeneracies $d_n = (n+1)^2$ as well as the energy spectrum $E_n = 2\hbar g\omega(n+1)$, do indeed agree with those determined in Ref.[13]. However, note how by having introduced the physical projector $\mathbb{E}$, all degeneracies d_n are readily obtained whatever the number d of particles involved in two dimensions, without the necessity of considering the subtle issues of gauge fixing, modular space and the spectrum of the ensuing Schrödinger equation, aspects which are all unavoidable in the standard approach adopted in Ref.[13].

6 The SO(2) Model: the SO(d) Valued Partition Function

The ever increasing degeneracies (46) with energy level beg for an understanding. A remark which goes part way towards a complete explanation is to notice that all d, two dimensional spherical harmonic oscillators share a common angular frequency ω. In other words, in addition to the SO(2) local gauge symmetry associated to the index $a = 1, 2$, the system also possesses a global SO(d) symmetry associated to rotations in the index $i = 1, 2, \cdots, d$. However, as we shall see, this property still does not account completely for all the observed degeneracies.

Therefore, all gauge invariant physical states must also fall into different representations of the global SO(d) symmetry, thereby explaining (to some extent) the obtained degeneracies. It should thus be possible to "tag" each of the physical states in terms both of the SO(d) representation to which it belongs and its quantum numbers under that symmetry. This is possible by extending the operator $x^{\hat{N}}\mathbb{E}$ to include the SO(d) transformation properties of physical states, namely by considering the SO(d) generators of the system.

Clearly, in terms of the cartesian or helicity quantum degrees of freedom, these $d(d-1)/2$ SO(d) generators are given by

$$\hat{L}_{ij} = i\hbar \left[\alpha_i^{a\dagger} \alpha_j^a - \alpha_j^{a\dagger} \alpha_i^a \right] = i\hbar \left[\alpha_i^{+\dagger} \alpha_j^+ + \alpha_i^{-\dagger} \alpha_j^- - \alpha_j^{+\dagger} \alpha_i^+ - \alpha_j^{-\dagger} \alpha_i^- \right] \quad , \tag{47}$$

and thus satisfy the algebra

$$\left[\hat{L}_{ij}, \hat{L}_{kl} \right] = -i\hbar \left[\delta_{ik} \hat{L}_{jl} - \delta_{il} \hat{L}_{jk} - \delta_{jk} \hat{L}_{il} + \delta_{jl} \hat{L}_{ik} \right] \quad . \tag{48}$$

Related to these expressions, the matrix representations of the SO(d) generators in the defining d-dimensional representation are given by

$$\left(T_{ij} \right)_{kl} = i\hbar \left[\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} \right] \quad , \tag{49}$$

so that in fact $\hat{L}_{ij} = \alpha^\dagger \cdot T_{ij} \cdot \alpha$, with the contraction operating on the indices $k, l = 1, 2, \cdots, d$ and a diagonal summation over the SO(2) gauge index $a = 1, 2$. Finite SO(d) transformations in the d-dimensional vector representation are then given by the $d \times d$ rotation matrices,

$$R_{kl}\left(\omega_{ij}\right) = \left(e^{-\frac{i}{2\hbar}\omega_{ij}T_{ij}}\right)_{kl} \quad , \tag{50}$$

where ω_{ij} are $d(d-1)/2$ angular variables parametrising the SO(d) group manifold.

Consequently (see Appendix A), creation operators transform as follows under the global SO(d) symmetry of the model,

$$e^{-\frac{i}{2\hbar}\omega_{ij}\hat{L}_{ij}}\,\alpha_i^{\pm\dagger}\,e^{\frac{i}{2\hbar}\omega_{ij}\hat{L}_{ij}} = \alpha_j^{\pm\dagger}\,R_{ji}\left(\omega_{ij}\right) \quad , \tag{51}$$

while for coherent states, one has

$$e^{-\frac{i}{2\hbar}\omega_{ij}\hat{L}_{ij}}\,|z_i^\pm> = |R_{ij}\left(\omega_{ij}\right)\,z_j^\pm> \quad . \tag{52}$$

Given these considerations, the generalisation of the operator $x^{\hat{N}}\boldsymbol{E}$ to an SO(d) valued operator, which "tags" physical states by their SO(d) representation quantum numbers, is simply defined by

$$x^{\hat{N}} e^{-\frac{i}{2\hbar}\omega_{ij}\hat{L}_{ij}}\,\boldsymbol{E} \quad . \tag{53}$$

For the same reasons as detailed in the previous Section, the trace of this operator provides the SO(d) valued partition function of physical states, while its diagonal matrix elements generate their SO(d) valued wave functions. Consequently, the operator (53) is the ideal tool for determining the SO(d) characterization of gauge invariant physical states.

Considering then the SO(d) valued partition function, one simply finds again,

$$\mathrm{Tr}\,x^{\hat{N}} e^{-\frac{i}{2\hbar}\omega_{ij}\hat{L}_{ij}}\,\boldsymbol{E} = \int_0^{2\pi}\frac{d\theta}{2\pi}\int\prod_{\pm,i}\frac{dz_i^\pm d\bar{z}_i^\pm}{\pi}\;<z_i^\pm|xe^{\pm i\theta}R_{ij}\left(\omega_{ij}\right)\,z_j^\pm>\;e^{-\frac{1}{2}(1-|x|)^2)|z|^2} \quad , \tag{54}$$

so that finally,

$$\mathrm{Tr}\,x^{\hat{N}} e^{-\frac{i}{2\hbar}\omega_{ij}\hat{L}_{ij}}\,\boldsymbol{E} = \int_0^{2\pi}\frac{d\theta}{2\pi}\,\frac{1}{\det\left[\delta_{ij} - xe^{i\theta}R_{ij}\left(\omega_{ij}\right)\right]\,\det\left[\delta_{ij} - xe^{-i\theta}R_{ij}\left(\omega_{ij}\right)\right]} \quad , \tag{55}$$

a result which generalises in a transparent way the one obtained previously for the partition function in (44).

The evaluation of this expression for an arbitrary choice of angular variables ω_{ij} would be quite involved. However, given our purpose of determining the SO(d) representations of physical states, it suffices to only consider a maximal commuting subalgebra among the generators $\hat{L}_{ij}$, namely the Cartan subalgebra[14]. Indeed, representations of compact semi-simple Lie algebras may be characterized by the Dynkin labels of the Dynkin diagram related to the Cartan subalgebra. In the case of the SO(d) group, two general classes have to be distinguished according to whether d is even or odd. Consequently, in order to proceed with the calculation of the SO(d) valued partition function restricted to the Cartan subalgebra, it proves useful to first consider the simple cases with $d = 1$ and $d = 2$, which will display the structure of the general solution.

The case $d = 1$ is trivial, since no global symmetry is then present, and the SO($d = 1$) valued partition function thus reduces to the simple partition function (44), namely,

$$d = 1: \qquad \mathrm{Tr}\,x^{\hat{N}}\boldsymbol{E} = \int_0^{2\pi}\frac{d\theta}{2\pi}\,\frac{1}{\left[1 - xe^{i\theta}\right]\left[1 - xe^{-i\theta}\right]} = \sum_{n=0}^{\infty}x^{2n} = \frac{1}{1 - x^2} \quad . \tag{56}$$

Consider now the $d=2$ case. Here again, the appearance of the global SO(2)=U(1) symmetry in the index $i=1,2$ suggests the introduction of helicity-like degrees of freedom, such as $\alpha_\pm^{\pm\dagger}$ and $\alpha_\pm^\pm$, as well as $z_\pm^\pm$, defined in terms of complex linear combinations of the cartesian creation and annihilation operators $\alpha_i^{\pm\dagger}$ and $\alpha_i^\pm$ $(i=1,2)$ (in the same way as were the latter helicity operators defined in terms of $\alpha_i^{a\dagger}$ and α_i^a $(a=1,2)$), with thus the associated helicity coherent states $|z_\pm^\pm>$ (see also Appendix A). Since the procedure should by now be clear, it is not spelled out explicitly here.

On the other hand, the SO(2) algebra is abelian with only one generator $\hat{L}_{12}$, thus defining in a trivial manner the Cartan subalgebra. This operator being given by (a summation over the upper index $\pm$ is implicit)

$$\hat{L}_{12}=i\hbar\left[\alpha_1^{\pm\dagger}\alpha_2^\pm-\alpha_2^{\pm\dagger}\alpha_1^\pm\right]=-\hbar\left[\alpha_+^{\pm\dagger}\alpha_+^\pm-\alpha_-^{\pm\dagger}\alpha_-^\pm\right]\quad,\tag{57}$$

its action on the coherent states $|z_\pm^\pm>$ is obviously

$$e^{-\frac{i}{2\hbar}\omega_{ij}\hat{L}_{ij}}\,|z_\pm^\pm>=|e^{\pm i\omega_{12}}z_\pm^\pm>\quad,\tag{58}$$

where the $\pm$ sign in the phase factor on the r.h.s. is of course correlated with that of the *lower* index of $z_\pm^\pm$.

Consequently, it should be clear that for $d=2$, by working in the $i=1,2$ helicity basis, the SO$(d=2)$ valued partition function (55) reduces to,

$$\mathrm{Tr}\,x^{\hat{N}}e^{-\frac{i}{2\hbar}\omega_{ij}\hat{L}_{ij}}\,\mathbb{E}=\int_0^{2\pi}\frac{d\theta}{2\pi}\,\frac{1}{\left[1-xe^{i(\theta+\omega_{12})}\right]\left[1-xe^{i(\theta-\omega_{12})}\right]\left[1-xe^{-i(\theta-\omega_{12})}\right]\left[1-xe^{-i(\theta+\omega_{12})}\right]}\quad.\tag{59}$$

Through a careful analysis of the series expansion of this integral in powers of x, one finally obtains

$$d=2:\qquad \mathrm{Tr}\,x^{\hat{N}}e^{-\frac{i}{2\hbar}\omega_{ij}\hat{L}_{ij}}\,\mathbb{E}=\sum_{n=0}^{\infty}x^{2n}\sum_{p=-n}^{n}\Big[(n+1)-|p|\Big]\,e^{2ip\omega_{12}}\quad.\tag{60}$$

This result thus establishes that for $d=2$, all $d_n=(n+1)^2$ physical states at energy level $E_n=2\hbar g\omega(n+1)$ $(n=0,1,\dots)$ fall into the one dimensional representations of the SO$(d=2)$=U(1) global symmetry of integer helicity $-n\le p\le n$, with a degeneracy $d(n,p)=(n+1-|p|)$ for each of these helicity representations in the index $i=1,2$. In particular, we have indeed that,

$$d=2:\qquad \sum_{p=-n}^{n}d(n,p)=(n+1)^2=d_n\quad,\quad n=0,1,2,\dots\quad.\tag{61}$$

In conclusion, the existence of the global SO$(d=2)$ symmetry does account for the degeneracies in the energy spectrum of physical states, though not completely, since the SO$(d=2)$ representations themselves possess some degree of degeneracy which still needs to be understood. As a matter of fact, remarks to that effect are made in the Conclusions, without attempting a general solution to this issue in the present paper.

Having solved the $d=2$ case, let us now turn first to the general situation when $d=2M$ is even. The Cartan subalgebra of the SO$(d=2M)$ global symmetry may then be described as follows. The $2M$ indices $i=1,2,\cdots,d(=2M)$ may be grouped into M consecutive pairs $(1,2)$, $(3,4)$, ..., $(2M-1,2M)$, to be labelled by an index $\mu=1,2,\cdots,M$ in the same order (thus the index μ refers to the pair $(i=2\mu-1,j=2\mu)$). Correspondingly, the Cartan subalgebra is spanned by the SO($d = 2M$) generators $\hat{L}_{ij}$ associated to this organisation in the (i,j) indices, with the identification,

$$\hat{L}_\mu = \hat{L}_{2\mu-1,2\mu} \quad , \quad \mu = 1,2,\ldots,M \quad , \tag{62}$$

thereby leading to M commuting generators in SO($d = 2M$), which is indeed the rank of the algebra of that group. In particular, finite group transformations restricted to the corresponding maximal torus in SO($d = 2M$) are generated by,

$$e^{-\frac{i}{2\hbar}\omega_{ij}\hat{L}_{ij}} = e^{-\frac{i}{\hbar}\omega_\mu \hat{L}_\mu} \quad , \tag{63}$$

with the following association in the angular parameters ω_μ and ω_{ij},

$$\omega_\mu = \omega_{2\mu-1,2\mu} \quad , \quad \mu = 1,2,\cdots,M \quad . \tag{64}$$

Clearly, through this construction we have achieved for each value of $\mu = 1,2,\cdots,M$, a situation identical to that developed previously for $d = 2$, to which the helicity techniques may again be applied. Hence, the Cartan subalgebra restricted SO($d = 2M$) valued partition function (55) simply reads,

$$\mathrm{Tr}\, x^{\hat{N}} e^{-\frac{i}{\hbar}\omega_\mu \hat{L}_\mu} \mathbb{E} = \int_0^{2\pi} \frac{d\theta}{2\pi} \prod_{\mu=1}^{M} \frac{1}{\left[1 - xe^{i(\theta+\omega_\mu)}\right]\left[1 - xe^{i(\theta-\omega_\mu)}\right]\left[1 - xe^{-i(\theta-\omega_\mu)}\right]\left[1 - xe^{-i(\theta+\omega_\mu)}\right]} \quad , \tag{65}$$

reducing finally to the expression,

$$\sum_{n^+_\mu, n^-_\mu, m^+_\mu, m^-_\mu = 0}^{\infty} x^{\sum_{\mu=1}^{M}(n^+_\mu + n^-_\mu + m^+_\mu + m^-_\mu)}\, e^{i\sum_{\mu=1}^{M}\omega_\mu(n^+_\mu - n^-_\mu - m^+_\mu + m^-_\mu)}\, \delta\left(\sum_{\mu=1}^{M}(n^+_\mu - n^-_\mu + m^+_\mu - m^-_\mu)\right) \quad . \tag{66}$$

In this power series expansion in x, the power of x determines the excitation level of a contributing gauge invariant physical state, while the Kronecker δ constraint corresponds to the left- and right-handed SO(2) gauge helicity matching condition (29) for that state. Finally, the SO($d = 2M$) properties of physical states are encoded in the phase factor involving the parameters ω_μ. The coefficients of these parameters determine the SO($d = 2M$) quantum numbers of these states, *i.e.* the Dynkin labels of the weight vectors associated to each one of these states as members of specific SO($d = 2M$) representations. In particular, the phase factors for highest weight states determine the Dynkin labels of the SO($d = 2M$) representations contributing to the partition function. In this manner, starting with the very highest weight state and identifying its SO($d = 2M$) descendants, and so on by recursion starting again from the remaining highest weight states, in principle it is possible to identify the content of SO($d = 2M$) representations appearing at each energy level in the physical spectrum, including their degeneracies. However, based on the $d = 2$ explicit solution, one expects that this classification does not exhaust completely the degeneracies of the energy spectrum, for which a deeper understanding is still called for in terms of a symmetry larger than SO($d = 2M$). Since such an analysis is deferred to later work, the complete characterization of the SO($d = 2M$) representations for physical states enscribed in (66) is not pursued further here.

Finally, turning to the case when $d = 2M + 1$ is odd, it should now be clear how to proceed. Simply, all $2M+1$ indices $i = 1,2,\cdots,(d = 2M+1)$ are again grouped into consecutive pairs $(1,2)$, $(3,4)$, $\cdots$, $(2M-1,2M)$, labelled by $\mu = 1,2,\cdots,M$ in the same order and thus leaving the odd index $i = 2M+1$ on its own. Correspondingly, the Cartan subalgebra is then obtained through the identification

$$\hat{L}_\mu = \hat{L}_{2\mu-1,2\mu} \quad , \quad \mu = 1,2,\cdots,M \quad , \tag{67}$$

hence M such commuting $SO(d = 2M + 1)$ generators, which is indeed the rank of the algebra of the group $SO(d = 2M+1)$. Clearly here again, each of the M pairs $(i = 2\mu-1, j = 2\mu)$ is identical to the $d = 2$ system, while the remaining index $i = 2M + 1$ may be viewed as the $d = 1$ system.

As a result, given the identification of $SO(d = 2M + 1)$ angular parameters $\omega_\mu = \omega_{2\mu-1,2\mu}$, the Cartan subalgebra restricted $SO(d = 2M + 1)$ valued partition function simply reads

$$\mathrm{Tr}\, x^{\hat{N}} e^{-\frac{i}{\hbar}\omega_\mu \hat{L}_\mu} \mathbb{E} =$$

$$\int_0^{2\pi} \frac{d\theta}{2\pi} \frac{1}{\left[1 - xe^{i\theta}\right]\left[1 - xe^{-i\theta}\right]} \prod_{\mu=1}^{M} \frac{1}{\left[1 - xe^{i(\theta+\omega_\mu)}\right]\left[1 - xe^{i(\theta-\omega_\mu)}\right]\left[1 - xe^{-i(\theta-\omega_\mu)}\right]\left[1 - xe^{-i(\theta+\omega_\mu)}\right]} \quad . \tag{68}$$

Again, this result may be expressed through a power series expansion in x, with phase factors determining the $SO(d = 2M + 1)$ quantum numbers of gauge invariant physical states. However, for the same reasons as those given for the $d = 2M$ case, this point shall not be pursued any further in this paper.

Nevertheless, the discussion of this Section has demonstrated unequivocally, that the physical projector approach allows for a complete characterization of all the symmetry properties of the physical spectrum of a gauge invariant theory. The next Section addresses the determination of the associated wave functions through the same calculational tools.

7 The SO(2) Model: Wave Functions of Physical States

As pointed out in (41), the determination of physical state wave functions entails simply the evaluation of the diagonal matrix elements of the operator $x^{\hat{N}}\mathbb{E}$, which are most straightforwardly obtained in the coherent state basis. Based on the considerations of the previous Section, this generating function for physical state wave functions may be extended by introducing a "tagging" in terms of representations of the $SO(d)$ global symmetry, for which it suffices to restrict $SO(d)$ group elements to the maximal torus defined by the Cartan subalgebra. In so doing, wave functions will appear accompanied in the generating function by the relevant phase factors which characterize their $SO(d)$ quantum numbers, thereby enabling a direct identification of the appropriate wave functions which otherwise would appear only through linear combinations due to the energy degeneracies. Nevertheless, degeneracies in the $SO(d)$ representations themselves could still lead to ambiguities in this identification of wave functions, if only in their proper normalisation[11], even though the determination of these degeneracies from the partition function along the lines outlined in the previous Section should allow to lift such ambiguities. Clearly, once identified, the symmetry beyond $SO(d)$ accounting completely for all such degeneracies would circumvent these issues altogether.

For the time being though, let us consider the following coherent state matrix elements,

$$< z_i^{\pm} | x^{\hat{N}} e^{-\frac{i}{2\hbar}\omega_{ij}\hat{L}_{ij}}\, \mathbb{E} | z_i^{\pm} > \quad , \tag{69}$$

where the finite $SO(d)$ group elements are restricted to the maximal torus defined by Cartan subalgebra. Given the developments of the previous Sections, it is clear that these matrix elements read as

$$< z_i^{\pm} | x^{\hat{N}} e^{-\frac{i}{2\hbar}\omega_{ij}\hat{L}_{ij}}\, \mathbb{E} | z_i^{\pm} > = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{-|z|^2} \prod_{\pm} e^{x e^{\pm i\theta} \bar{z}_i^{\pm} R_{ij}(\omega_{ij}) z_j^{\pm}} \quad . \tag{70}$$

[11] Indeed, the generating function should provide not only the wave functions, but also their proper normalisation.

Let us first assume $d = 2M$ to be even. Given the Cartan subalgebra of SO($d = 2M$) described in the previous Section, and the choice of the corresponding helicity coordinate basis in the index $i = 1, 2, \cdots, (d = 2M)$, (70) becomes,

$$\int_0^{2\pi} \frac{d\theta}{2\pi} e^{-|z|^2} \prod_{\pm} \left\{ \prod_{\mu=1}^{M} e^{xe^{\pm i\theta}\left[e^{i\omega_\mu}|z^\pm_{\mu+}|^2 + e^{-i\omega_\mu}|z^\pm_{\mu-}|^2\right]} \right\} \quad , \tag{71}$$

where,

$$z^\pm_{\mu+} = \frac{1}{\sqrt{2}}\left[z^\pm_{2\mu-1} - iz^\pm_{2\mu}\right] \qquad z^\pm_{\mu-} = \frac{1}{\sqrt{2}}\left[z^\pm_{2\mu-1} + iz^\pm_{2\mu}\right] \quad . \tag{72}$$

Upon expansion in x, one thus obtains for the Cartan subalgebra restricted SO($d = 2M$) valued generating function of physical state wave functions,

$$\sum_{n=0}^{\infty} x^{2n} \frac{e^{-|z|^2}}{(n!)^2} \left\{ \sum_{\mu,\nu=1}^{M} \left(e^{i\omega_\mu}|z^+_{\mu+}|^2 + e^{-i\omega_\mu}|z^+_{\mu-}|^2\right)\left(e^{i\omega_\nu}|z^-_{\nu+}|^2 + e^{-i\omega_\nu}|z^-_{\nu-}|^2\right) \right\}^n \quad . \tag{73}$$

Note that in this last expression, the term in curly brackets (raised to the n-th power) may be expressed as

$$\sum_{\mu,\nu=1}^{M} \left(e^{i(\omega_\mu+\omega_\nu)}|z^+_{\mu+}z^-_{\nu+}|^2 + e^{i(\omega_\mu-\omega_\nu)}|z^+_{\mu+}z^-_{\nu-}|^2 + e^{-i(\omega_\mu-\omega_\nu)}|z^+_{\mu-}z^-_{\nu+}|^2 + e^{-i(\omega_\mu+\omega_\nu)}|z^+_{\mu-}z^-_{\nu-}|^2\right) \quad . \tag{74}$$

Similarly in the case $d = 2M + 1$, (70) is given by,

$$\int_0^{2\pi} \frac{d\theta}{2\pi} e^{-|z|^2} \prod_{\pm} \left\{ e^{xe^{\pm i\theta}|z^\pm_{2M+1}|^2} \prod_{\mu=1}^{M} e^{xe^{\pm i\theta}\left[e^{i\omega_\mu}|z^\pm_{\mu+}|^2 + e^{-i\omega_\mu}|z^\pm_{\mu-}|^2\right]} \right\} \quad , \tag{75}$$

which leads to the series representation,

$$\sum_{n=0}^{\infty} x^{2n} \frac{e^{-|z|^2}}{(n!)^2} \times$$

$$\times \left\{ \left(\sum_{\mu=1}^{M} (e^{i\omega_\mu}|z^+_{\mu+}|^2 + e^{-i\omega_\mu}|z^+_{\mu-}|^2) + |z^+_{2M+1}|^2 \right) \left(\sum_{\mu=1}^{N} (e^{i\omega_\nu}|z^-_{\nu+}|^2 + e^{-i\omega_\nu}|z^-_{\nu-}|^2) + |z^-_{2M+1}|^2 \right) \right\}^n \quad . \tag{76}$$

As in (74), the term in curly brackets may be expanded as follows,

$$\sum_{\mu,\nu=1}^{M} \left(e^{i(\omega_\mu+\omega_\nu)}|z^+_{\mu+}z^-_{\nu+}|^2 + e^{i(\omega_\mu-\omega_\nu)}|z^+_{\mu+}z^-_{\nu-}|^2 + e^{-i(\omega_\mu-\omega_\nu)}|z^+_{\mu-}z^-_{\nu+}|^2 + e^{-i(\omega_\mu+\omega_\nu)}|z^+_{\mu-}z^-_{\nu-}|^2\right) +$$

$$+ \sum_{\mu=1}^{M} \left(e^{i\omega_\mu}|z^+_{\mu+}z^-_{2M+1}|^2 + e^{-i\omega_\mu}|z^+_{\mu-}z^-_{2M+1}|^2 + e^{i\omega_\mu}|z^+_{2M+1}z^-_{\mu+}|^2 + e^{-i\omega_\mu}|z^+_{2M+1}z^-_{\mu-}|^2\right) +$$

$$+ |z^+_{2M+1}z^-_{2M+1}|^2 \quad . \tag{77}$$

Recalling the arguments leading to (41), the modulus squared wave functions of gauge invariant physical states at energy level E_n are thus obtained by expanding the n-th power of the sums

(74) and (77). It is clear that these expansions lead indeed to sums of modulus squared products of polynomials in the coherent state labels $z_i^{\pm}$, thereby also enabling a direct identification of the combination of Fock state basis vectors associated to these physical states, including their proper normalisation. The appropriate linear combinations of these terms which are to be associated to given physical states are (partly) determined on the basis of their SO(d) quantum numbers "tagged" by the Dynkin label phase factors $e^{\pm i\omega_\mu}$. In the present analysis, only the remaining degeneracies in the SO(d) representations themselves—rather than their state content—could lead to some ambiguities in these identifications, and if these degeneracies are known on basis of the SO(d) valued partition function, the ambiguities would only be in the normalisation of the states which then has to be calculated separately. However, as was pointed out previously, the identification of the complete symmetry accounting for all observed degeneracies should lift any such remaining ambiguity, including the proper normalisation of all physical states.

The general structure of the expressions (74) and (77) is also quite interesting. Indeed, each modulus squared term involves the product of two $z_i^{\pm}$ factors whose upper indices in $a = 1, 2$ are opposite in the helicity basis. This fact makes the SO(2) gauge invariance of the associated states indeed obvious, since the corresponding SO(2) charge is then identically vanishing. Similarly, the structure in the lower index $i = 1, 2, \cdots, d$ of the products of the $z_i^{\pm}$ factors determines directly the corresponding SO(d) symmetry properties under transformations in the maximal torus generated by the Cartan subalgebra. Hence, upon expansion of the n-th power of (74) and (77) leading to specific binomial coefficients directly related to the normalisation of their wave functions, the SO(d) symmetry properties of physical states may be directly read off the combination of lower indices appearing in a given product of the $z_i^{\pm}$ factors.

It is useful to consider the cases $d = 1$ and $d = 2$ in some detail. When $d = 1$, the generating function of physical state wave functions is simply

$$< z^{\pm}|x^{\hat{N}}\mathbb{E}|z^{\pm} >= \sum_{n=0}^{\infty}\frac{x^{2n}}{(n!)^2}\,|z^+z^-|^{2n}\,e^{-|z^+|^2-|z^-|^2} \quad . \tag{78}$$

Consequently, the helicity coherent state wave functions of the single physical state at each energy level $E_n = \hbar g\omega(2n + 1)$ are given by

$$< E_n|z^{\pm} >= \frac{1}{n!}\,\left(z^+z^-\right)^n\,e^{-\frac{1}{2}\left[|z^+|^2+|z^-|^2\right]} \quad , \tag{79}$$

thus corresponding to the following orthonormalised Fock states

$$|E_n >= \frac{1}{n!}\left(\alpha^{+\dagger}\right)^n\,\left(\alpha^{-\dagger}\right)^n\,|0> \quad . \tag{80}$$

Of course, in the case $d = 1$, these results could have been obtained directly from the energy spectrum (28) and the level matching condition (29).

In the $d = 2$ case, the physical wave function generating function is given by[12]

$$< z_{\pm}^{\pm}|x^{\hat{N}}e^{-\frac{1}{2}\omega_{ij}\hat{L}_{ij}}\mathbb{E}|z_{\pm}^{\pm} >= \sum_{n=0}^{\infty}\frac{x^{2n}}{(n!)^2}\,e^{-|z|^2}\times$$

$$\times\left\{e^{2i\omega_{12}}|z_+^+z_+^-|^2 + \left|\frac{1}{\sqrt{2}}\left(z_+^+z_-^- + z_-^+z_+^-\right)\right|^2 + \left|\frac{1}{\sqrt{2}}\left(z_+^+z_-^- - z_-^+z_+^-\right)\right|^2 + e^{-2i\omega_{12}}|z_-^+z_-^-|^2\right\}^n \quad , \tag{81}$$

[12] The single value of the index $\mu = 1$ is this case is not displayed, while $|z|^2$ stands for the sum $\sum_{\pm}(|z_+^{\pm}|^2 + |z_-^{\pm}|^2)$.

where the two middle terms in (74) with vanishing Dynkin label have been expressed as symmetric and antisymmetric combinations under the exchange of the two harmonic oscillators $i = 1$ and $i = 2$. In this way, the identification of physical states and of their coherent state wave functions is straightforward enough, without the possibility of any ambiguity[13]. Thus, for example, at the first two energy levels $n = 0$ and $n = 1$, the properly normalised physical wave functions are simply

$$
\begin{aligned}
E_0 = 2\hbar g\omega \quad &: \quad < E_0, p = 0|z^{\pm}_{\pm} >= e^{-\frac{1}{2}\left[|z^+_+|^2+|z^+_-|^2+|z^-_+|^2+|z^-_-|^2\right]} \quad , \\
E_1 = 4\hbar g\omega \quad &: \quad < E_1, p = \pm 1|z^{\pm}_{\pm} >= \left(z^+_\pm z^-_\pm\right)\, e^{-\frac{1}{2}\left[|z^+_+|^2+|z^+_-|^2+|z^-_+|^2+|z^-_-|^2\right]} \quad , \\
&\quad\ \ < E_1, p = 0, \sigma = \pm 1|z^{\pm}_{\pm} >= \frac{1}{\sqrt{2}}\left(z^+_+ z^-_- \pm z^+_- z^-_+\right)\, e^{-\frac{1}{2}\left[|z^+_+|^2+|z^+_-|^2+|z^-_+|^2+|z^-_-|^2\right]} \quad ,
\end{aligned}
\tag{82}
$$

corresponding to the Fock states

$$
\begin{aligned}
E_0 = 2\hbar g\omega \quad &: \quad |E_0, p = 0 >= |0> \quad , \\
E_1 = 4\hbar g\omega \quad &: \quad |E_1, p = \pm 1 >= \alpha^{+\,\dagger}_{\pm}\,\alpha^{-\,\dagger}_{\pm}\,|0> \quad , \\
&\quad\ \ |E_1, p = 0, \sigma = \pm 1 >= \frac{1}{\sqrt{2}}\left(\alpha^{+\,\dagger}_{+}\,\alpha^{-\,\dagger}_{-} \pm \alpha^{+\,\dagger}_{-}\,\alpha^{-\,\dagger}_{+}\right)|0> \quad ,
\end{aligned}
\tag{83}
$$

where $p = 0, \pm 1$ is the SO($d = 2$) helicity label appearing in (60), while the additional index $\sigma = \pm 1$ distinguishes the permutation properties under the exchange in the indices $i = 1$ and $i = 2$ of the otherwise two degenerate states with $p = 0$. Clearly, similar considerations apply to all levels of the physical energy spectrum, thus including representations of the permutation symmetry group in the index $i = 1, 2$ to distinguish otherwise degenerate SO($d = 2$)=U(1) representations.

Given the helicity coherent wave functions of all physical states, it is possible to also determine their position wave functions for the original degrees of freedom q^a_i, which will involve only gauge invariant combinations of these degrees of freedom. In the $d = 2$ case, these wave functions could be compared to those determined in Ref.[13] over modular space, which is parametrised in terms of specific gauge invariant combinations of the original degrees of freedom q^a_i. Even though this worthwhile exercise is not attempted here, there is no doubt that identical results will be derived, involving the appropriate special functions[13]. Nevertheless, note how, through the physical projector $\mathbb{E}$, the coherent state basis allows for simple polynomial wave functions for gauge invariant states (up to the characteristic gaussian factor $e^{-|z|^2/2}$), in direct correspondence with the Fock states spanning the physical spectrum.

8 The SO(3) Gauge Invariant Model

The next simplest generalisation of the SO(2) model considered so far, involves dynamics invariant under the non abelian gauge symmetry group SO(3), determined from the Lagrange function of real degrees of freedom q^a_i and λ^a $(a = 1, 2, 3; i = 1, 2, \cdots, d)$,

$$L = \frac{1}{2g^2}\left[\dot{q}^a_i + \epsilon^{abc}\lambda^b q^c_i\right]^2 - V(q^a_i) \quad , \tag{84}$$

with

$$V(q^a_i) = \frac{1}{2}\omega^2\, q^a_i q^a_i \quad . \tag{85}$$

[13]The real reason for this fact will be made clear in the Conclusions.

Here again, g and ω are arbitrary parameters with appropriate physical dimensions, and ϵ^{abc} is the three dimensional antisymmetric tensor such that $\epsilon^{123} = +1$. As a matter of fact, λ^a correspond to the single time component of the non abelian SO(3) gauge field transforming under the three dimensional adjoint representation of SO(3), as do the matter fields q_i^a, while ϵ^{abc} are the structure coefficients of the associated Lie algebra[14].

Consequently, given the choice of SO(3) invariant potential in (85), the Lagrange function (84) describes the motion of d three dimensional spherically symmetric harmonic oscillators, whose total angular momentum vector must identically vanish at all times. In particular, SO(3) gauge transformations of the coordinates q_i^a and of the gauge variables λ^a are of the form

$$q'^a_i = U^{ab} q_i^b \quad , \quad \lambda'^a = U^{ab}\lambda^b - \frac{1}{2}\epsilon^{abc} U^{bd} \dot{U}^{cd} \quad , \tag{86}$$

where U^{ab} is the 3×3 rotation matrix defined in (157) of Appendix B, whose Euler angles ψ, θ and ϕ are arbitrary time dependent functions.

Clearly, in this case as well, the gauge variables λ^a may be gauged away entirely, leading in the $\lambda^a = 0$ gauge to equations of motion which are those of d three dimensional spherically symmetric harmonic oscillators of identical angular frequencies, constrained to possess an identically vanishing total angular momentum vector.

Similarly, the Hamiltonian analysis of the classical system goes through as in the SO(2) case. The first-class Hamiltonian H_0 and gauge constraints ϕ^a are given by

$$H_0 = \frac{1}{2} g^2 p_i^a p_i^a + \frac{1}{2}\omega^2 q_i^a q_i^a \quad , \quad \phi^a = \epsilon^{abc} q_i^b p_i^c \quad , \tag{87}$$

where p_i^a are the momenta conjugate to the coordinates q_i^a, while the total fundamental Hamiltonian reads

$$H = H_0 - \lambda^a \phi^a \quad , \tag{88}$$

the gauge variables λ^a being indeed (opposite to) the Lagrange multipliers for the first-class constraints ϕ^a. The latter, corresponding to the cartesian components of the total angular momentum vector, generate the SO(3) Lie algebra

$$\{\phi^a, \phi^b\} = \epsilon^{abc}\phi^c \quad , \tag{89}$$

leave the first-class Hamiltonian invariant, $\{H_0, \phi^a\} = 0$, and induce the following infinitesimal SO(3) gauge transformations on phase space, including those of the Lagrange multipliers λ^a,

$$\delta_\eta q_i^a = \epsilon^{abc}\eta^b q_i^c \quad , \quad \delta_\eta p_i^a = \epsilon^{abc}\eta^b p_i^c \quad , \quad \delta_\eta \lambda^a = -\dot{\eta}^a + \epsilon^{abc}\eta^b\lambda^c \quad , \tag{90}$$

given arbitrary time dependent (infinitesimal) functions η^a. Again, these transformations exponentiate to finite gauge transformations identical to those (86) of the Lagrangian formulation.

It should be clear how in the gauge $\lambda^a = 0$, the Hamiltonian equations of motion are simply,

$$\dot{q}_i^a = g^2 p_i^a \quad , \quad \dot{p}_i^a = -\omega^2 q_i^a \quad , \tag{91}$$

whose solutions are subjected to the constraints,

$$\phi^a = \epsilon^{abc} q_i^b p_i^c = 0 \quad . \tag{92}$$

[14]With the generators in the adjoint representation given by $(T^a)^{bc} = -i\epsilon^{abc}$.

9 The SO(3) Model: Gauge Invariant Partition Functions

The canonical quantisation of the SO(3) model is straightforward enough, and details are not given since they are similar to those for the SO(2) model. In particular, the excitation level operator $\hat{N} = \alpha_i^{a\dagger}\alpha_i^a$ extends the range of the index $a = 1, 2, 3$ in terms of creation and annihilation operators defined as previously, while the quantum first-class Hamiltonian $\hat{H}_0$ reads,

$$\hat{H}_0 = \hbar g\omega \left[\hat{N} + \frac{3}{2}d\right] \quad . \tag{93}$$

As for the SO(2) model, the physical spectrum of gauge invariant states, including their degeneracies, is obtained from the partition function,

$$\mathrm{Tr}\, x^{\hat{N}} \mathbb{E} \quad , \tag{94}$$

where x is a complex parameter, possibly constrained by $|x| < 1$, and $\mathbb{E}$ is the physical projection operator. In the present model, the latter operator is defined by

$$\mathbb{E} = \int_{\mathrm{SO}(3)} d\mu(\theta^a) e^{\frac{i}{\hbar}\theta^a \hat{\phi}^a} = \frac{1}{8\pi^2} \int_0^{2\pi} d\psi \int_0^{\pi} d\theta \, \sin\theta \int_0^{2\pi} d\phi \, e^{\frac{i}{\hbar}\psi\hat{\phi}^3} e^{\frac{i}{\hbar}\theta\hat{\phi}^1} e^{\frac{i}{\hbar}\phi\hat{\phi}^3} \quad , \tag{95}$$

where $\hat{\phi}^a = \epsilon^{abc}\hat{q}_i^b\hat{p}_i^c$ are the SO(3) generators of the quantised system. Indeed, in terms of the Euler angle parametrisation given in Appendix B, the integration measure $d\mu(\theta^a)$ is the SO(3) invariant Haar measure, while the product of transformations $e^{\frac{i}{\hbar}\psi\hat{\phi}^3} e^{\frac{i}{\hbar}\theta\hat{\phi}^1} e^{\frac{i}{\hbar}\phi\hat{\phi}^3}$ gives the corresponding SO(3) transformation, which could also be expressed as $e^{\frac{i}{\hbar}\theta^a\hat{\phi}^a}$ in terms of specific angles θ^a functions of the Euler angles ψ, θ and ϕ.

Based on the arguments developed for the SO(2) model using coherent states, it should be clear that the above partition function for the SO(3) model is simply given by[15]

$$\mathrm{Tr}\, x^{\hat{N}} \mathbb{E} = \int_{\mathrm{SO}(3)} d\mu(\theta^a) \, \frac{1}{\left[\det\left(\delta^{ab} - xU^{ab}(\theta^a)\right)\right]^d} \quad , \tag{96}$$

whose explicit evaluation leads to the following results[16].

When $d = 1$, one finds again

$$\mathrm{Tr}\, x^{\hat{N}} \mathbb{E} = \sum_{n=0}^{\infty} x^{2n} = \frac{1}{1 - x^2} \quad . \tag{97}$$

Thus in this case, the physical energy spectrum is given by

$$E_n = \hbar g\omega \left(2n + \frac{3}{2}\right) \quad , \quad n = 0, 1, 2, \ldots \quad , \tag{98}$$

and is free of any degeneracy at any level n,

$$d_n = 1 \quad , \quad n = 0, 1, 2, \ldots \quad . \tag{99}$$

[15] The expression of the relevant determinant is given in Appendix B.

[16] The case $d = 1$ needs to be considered separately from $d \geq 2$. The integration over one of the two angles ψ or ϕ is trivial since they are summed, $\psi + \phi$, while the integration over the remaining one, say ϕ, is best done through the change of variable $u = \tan(\phi/2)$.

When $d \geq 2$, one has

$$\mathrm{Tr}\, x^{\hat{N}} \mathbb{E} = \frac{(1+x)^{d-2}}{(1-x^2)^{3(d-1)}} \sum_{n=0}^{d-2} \frac{(d-2)!}{n!\ (d-2-n)!} \frac{(2n)!}{n!\ (n+1)!} x^n (1-x)^{2(d-2-n)} \quad , \tag{100}$$

thus clearly displaying the existence of degeneracies at all non trivial physical energy levels, starting with the first gauge invariant excitation level $\hat{N} = 2$ or $n = 1$ for which $d_{n=1} = \frac{1}{2} d(d+1)$. For example for $d = 2$, one finds results which of course agree with those of Ref.[13],

$$\mathrm{Tr}\, x^{\hat{N}} \mathbb{E} = \frac{1}{(1-x^2)^3} = \sum_{n=0}^{\infty} \frac{1}{2} (n+1)(n+2)\, x^{2n} \quad , \tag{101}$$

showing that the physical energy spectrum is then given by,

$$E_n = \hbar g \omega\, (2n+3) \quad , \quad n = 0, 1, 2, \dots \quad , \tag{102}$$

with the following degeneracies at each level n,

$$d_n = \frac{1}{2} (n+1)(n+2) \quad , \quad n = 0, 1, 2, \dots \quad . \tag{103}$$

In the case $d = 3$, one finds

$$\mathrm{Tr}\, x^{\hat{N}} \mathbb{E} = \frac{(1+x^3)}{(1-x^2)^6} = \sum_{n=0}^{\infty} \frac{(n+5)!}{5!\ n!} x^{2n} (1+x^3) \quad , \tag{104}$$

establishing that the physical energy spectrum is separated into two classes with identical degeneracies, for all values $n = 0, 1, 2, \dots$,

$$\begin{aligned} E_n^{(1)} &= \hbar g \omega \left(2n + \tfrac{9}{2} \right) \quad , \quad d_n^{(1)} = \tfrac{1}{120} (n+1)(n+2)(n+3)(n+4)(n+5) \quad , \\ E_n^{(2)} &= \hbar g \omega \left(2n + \tfrac{15}{2} \right) \quad , \quad d_n^{(2)} = \tfrac{1}{120} (n+1)(n+2)(n+3)(n+4)(n+5) \quad . \end{aligned} \tag{105}$$

The appearance of this new feature is related to the fact that starting with $d = 3$, new SO(3) gauge invariant combinations of the creation operators $\alpha_i^{a\dagger}$ become possible using the SO(3) invariant structure coefficients ϵ^{abc}, in addition to the only other SO(3) invariant tensor δ^{ab} which may be used whatever the value of d.

Similarly, for still larger values of d, the structure of the physical energy spectrum and its degeneracies becomes even richer and still more involved, in a manner which deserves to be fully understood. In particular, the SO(d) global symmetry of the model goes some way in accounting for the observed degeneracies, but here again, some larger symmetry still to be identified must be at work. Considering nevertheless the SO(d) Cartan subalgebra valued partition function, it should be clear that this quantity is given by the following integral over the SO(3) manifold,

$$\mathrm{Tr}\, x^{\hat{N}} e^{-\frac{1}{2}\omega_{ij} \hat{L}_{ij}} \mathbb{E} = \int_{\mathrm{SO}(3)} d\mu(\theta^a) \frac{1}{\det \left[\delta^{ab} \delta_{ij} - x U^{ab}(\theta^a) R_{ij}(\omega_{ij}) \right]} \quad , \tag{106}$$

with the same notations and operators as those introduced already for the SO(2) model. Refraining from establishing here general results for arbitrary values of d, let us concentrate on the $d = 2$ case, which carries with it the basic structure for a general solution as was already observed for the SO(2) model. Working in the helicity basis for the index $i = 1, 2$, an explicit calculation then finds

$$\mathrm{Tr}\, x^{\hat{N}} e^{-\frac{i}{\hbar}\omega_{ij}\hat{L}_{ij}} \mathbb{E} = \frac{1}{(1-x^2)(1-x^2 e^{2i\omega_{12}})(1-x^2 e^{-2i\omega_{12}})} = \sum_{n=0}^{\infty} x^{2n} \sum_{p=-n}^{n} d(n,p) e^{2ip\omega_{12}} \quad , \tag{107}$$

with the SO($d = 2$) helicity degeneracies

$$d(n,p) = E\left[\frac{n+|p|}{2}\right] - |p| + 1 \quad , \quad -n \le p \le n \quad , \quad n = 0, 1, 2, \ldots \quad , \tag{108}$$

where $E[\rho]$ is the integer part of a real variable ρ, such that indeed,

$$\sum_{p=-n}^{n} d(n,p) = \frac{1}{2}(n+1)(n+2) \quad , \quad n = 0, 1, 2, \ldots \quad . \tag{109}$$

Even though possible at this stage, the extension of these results to larger values of d is to be addressed in later work, which should also include the construction of the generating function for coherent state wave functions of all gauge invariant physical states. Indeed, once identified, and with the help of the appropriate group valued partition and generating functions for physical states, the complete symmetry accounting for all degeneracies of the physical spectrum, including those of the SO(d) representations themselves, should in principle render the analysis of all these issues more transparent. This specific aspect of this class of models is deferred to later work. Nevertheless, the analysis of the present paper has clearly demonstrated the advantages in using the physical projection operator in conjunction with techniques of coherent states, to disentangle the physical spectrum of gauge invariant systems.

10 Conclusions

In this paper, the physical projector operator constructed in Ref.[7] for general constrained systems, is applied to a particular class of gauge invariant quantum mechanical systems. In contradistinction to more standard approaches which require gauge fixing and generally also additional degrees of freedom, by working only with the initial variables of a system, the physical projector approach to gauge invariant systems avoids these requirements altogether, and thereby also the ensuing delicate issues of possible Gribov problems and the characterization of modular space. Nevertheless, it provides both a manifestly covariant as well as a consistent unitary description of the dynamics of gauge invariant physical states. As the present analysis has demonstrated even within the context of relatively simple models, when conjugated with techniques of coherent states, the physical projector approach offers many advantages over more traditional ones. For instance, at least for the models considered, it reduces the determination of their physical spectrum to an exercise in the combinatorics of compact Lie group representation theory and in the integration over group manifolds of characters of these representations[17].

Even though only two types of models were considered explicitly in the paper, namely one with the abelian gauge group SO(2) and the other with the non abelian gauge group SO(3), these

[17] Indeed, in the general case, the evaluation of a quantity such as (96), and more specifically of the associated generating function of physical state wave functions, involves precisely the group integration of characters of representations, when expanding these expressions in terms of traces of powers of the matrices involved.

examples belong to quite general classes of systems whose dynamics is described by Lagrange functions of the form[18],

$$L = \frac{1}{2g^2}\left[\dot{q}_i^\alpha + i\lambda^a\left(T^a\right)^{\alpha\beta} q_i^\beta\right]^2 - V(q_i^\alpha) \quad . \tag{110}$$

Here, $q_i^\alpha(t)$ ($\alpha = 1, 2, \cdots, r;\ i = 1, 2, \cdots, d$) are a collection of d scalar matter fields transforming in some r-dimensional real representation of a gauge group G, whose generators are (dim G) hermitian $r \times r$ matrices T^a ($a = 1, 2, \cdots, \dim G$). The gauge field has as single time component the real quantities $\lambda^a(t)$ ($a = 1, 2, \cdots, \dim G$). Finally, $V(q_i^\alpha)$ is some specific G-invariant potential for the matter fields q_i^α.

The models considered in the present paper correspond to the gauge groups SO(2) and SO(3), with the d matter fields in the defining representation of these groups, and a harmonic potential of identical angular frequency for all d matter fields. More generally, the gauge group SO(N) with d matter fields in the N-dimensional representation could be considered[19]. Another possibility is to choose the d matter fields in the (dim G)-dimensional adjoint representation of the gauge group, in which case the matrices T^a are given by the structure coefficients f^{abc} of the associated Lie algebra, $(T^a)^{bc} = -if^{abc}$. Note that the $G =$SO(3) model of the paper belongs to this class, while for an arbitrary gauge group G, the case with $d = 1$ has already been considered in Refs.[9, 11].

Clearly, the approach of the present paper may be applied to many classes of such models, including those based on the exceptional Lie algebras (starting for example with G_2 whose representations are all real). Beyond those cases, the analysis of 1+1 dimensional non abelian pure Yang-Mills theories (possibly also coupled to bosonic or fermionic matter fields) may also be addressed along similar lines, by compactifying, for example, the space dimension onto a circle (this type of model is also considered in Ref.[13] for SO(3)). Indeed, the dimensional reduction of such models to 0+1 dimensions belongs precisely to the general class of systems defined by (110). Likewise, the compactifications to 1+1 and 0+1 dimensions of models motivated by the dualities of M-theory[12] could provide another fertile field for the physical projector approach. Finally, the potentialities offered by this approach are there to be explored beyond 1+1 dimensional Yang-Mills systems, beginning with the fascinating case of 2+1 dimensional Chern-Simons theories[15], both for compact and non-compact Lie groups, the latter case being related to theories of gravity[16].

The general programme just outlined should also help unravel, and benefit from, those issues left open in the present work, related to the genuine reasons for the observed degeneracies in the physical energy spectrum of the SO(2) and SO(3) models. As explained in Appendix A in the case of the d-dimensional spherical harmonic oscillator, the classical symmetries of a system may be enhanced at the quantum level into dynamical symmetries. Indeed, for the general class of models defined by (110), when the potential $V(q_i^\alpha)$ is chosen to be that of $r \times d$ harmonic oscillators of identical angular frequency ω, the quantised system possesses a *global* dynamical symmetry based on the unitary group U(rd)=U(1)$_N$×SU(rd) acting on the matter fields q_i^α, independently of the existence of the local gauge symmetries based on the group G. The U(1)$_N$ symmetry is generated by the total excitation level operator $\hat{N}$, and thus acts in a trivial manner on the space of states. On the other hand, the SU(rd) symmetry possesses the subgroup SU(r)×SU(d), where the first factor acts on the α indices of the coordinates q_i^α, while the second factor acts on the i indices. The dynamical symmetry SU(d) thus enhances at the quantum level the global SO(d) symmetry of the classical system, the latter being embedded into the former. Similarly, the *local* gauge symmetry

[18]Such Lagrange function may be extended to arbitrary complex representations of the gauge group as well, in an obvious way.

[19]The case $d = 1$ having already been discussed in Refs.[8, 9], whose results agree of course with those of the present analysis for SO($N = 2$) and SO($N = 3$) when $d = 1$.

group G is a subgroup of the *global* dynamical symmetry SU(r), and it is by understanding how G is embedded into SU(r) that the global SU(d) symmetry properties of gauge invariant physical states can be determined.

Indeed, since the creation operators $\alpha_i^{\alpha\dagger}$ all commute with one another and transform in the fundamental rd-dimensional representation of SU(rd), at the excitation level $\hat{N} = n$, all states of the quantised system fall into the n times totally symmetric representation of SU(rd). Reducing that representation to the subgroups SU(r)×SU(d) and then G×SU(d), and retaining only those representations which are singlets under the gauge symmetry group G, leads to the identification of all physical states at excitation level $\hat{N} = n$ including their SU(d) symmetry properties. Consequently, the symmetry accounting for all degeneracies in the physical energy spectrum is the dynamical group[20] SU(d). Hence, rather than the classical SO(d) symmetry, it is the quantum dynamical SU(d) symmetry, through its Cartan sublagebra, which must be used in the partition and wave function generating functions to "tag" physical states and their global symmetry properties, possibly also including a "tagging" for the SU(d) Casimir operators in order to easily disentangle the SU(d) representation content of the physical spectrum, and in particular the wave functions of the highest weight states and their descendants in each of the SU(d) representations. By combining the insight provided by this dynamical symmetry with the advantages of the physical projector and coherent state techniques, it thus appears possible to give a completely structured characterization of the physical spectrum of the class of models defined by (110).

In the case of the SO(2) and SO(3) models of the present paper, all physical states thus fall into representations of SU(d). For example, at the first non trivial physical excitation level $\hat{N} = 2$, these representations are as follows. When $G =$SO(2), one finds the two index symmetric and antisymmetric SU(d) representations, of dimensions $d(d+1)/2$ and $d(d-1)/2$ respectively, hence indeed a total of d^2 states. When $G =$SO(3), one finds only the two index symmetric SU(d) representation, indeed a total of $d(d+1)/2$ states.

The dynamical symmetry also helps disentangle the wave functions from the generating function. To illustrate this point, consider simply the SO(2) model with $d = 2$, for which the generating function is given in (81). How is one to associate the $(n+1)(n+2)(n+3)/6$ terms stemming from the n-th power of the sum of four terms to the $(n+1)^2$ physical states at the excitation level $\hat{N} = 2n$? The answer to this question is not immediate from the helicity quantum number $-n \leq p \leq n$ associated to the global SO($d = 2$) symmetry, but it is from the SU($d = 2$) dynamical symmetry[21]. The global SU(4) dynamical symmetry in this case being of rank 3, its Cartan sublagebra is spanned by that of the subgroup SU($r = 2$)×SU($d = 2$), which is of rank 2, and an intertwining operator $\hat{S}$ coupling these two SU(2) factors. Moreover, the embedding of the gauge group SO(2) and of the global symmetry group SO($d = 2$) into each of these SU(2) factors may be chosen such that their generators $\hat{\phi}$ and $\hat{L}_{12}$ coincide with the respective Cartan subalgebra T_3 generators both in SU($r = 2$) and in SU($d = 2$). Consequently, physical states are such that their T_3 eigenvalue for SU($r = 2$) vanishes while their T_3 eigenvalue for SU($d = 2$) corresponds to the SO($d = 2$) helicity quantum number p. Thus in addition to their energy (or excitation level) eigenvalue, physical states are labelled by SU($d = 2$) quantum numbers, namely a "spin" value T as well as the $T_3 = p$ projection in SU($d = 2$), *i.e.* $|E_n, T, T_3>$. At physical energy level E_n, the "spin" content of states is $T = 0, 1, 2, \cdots, n$, with each of these representations appearing only once, establishing that the dynamical SU($d = 2$) symmetry does indeed account completely for all de-

[20]When the variables q_i^α transform under a complex representation of the gauge group G, this conclusion must be adapted appropriatedly.

[21]Note that the dynamical SU(d) symmetry also includes the permutation symmetry group S_d acting on the d indices i, a discrete symmetry which was used to lift degeneracies in the SO(2) case with $d = 2$ (see Sect.7).

generacies in the physical spectrum of the model. And the same conclusion also enables the correct identification of the wave functions for all states from the generating function (81), including their normalisation. At energy level E_n, one way is to start from the highest spin state with $T = n$ and $T_3 = n$, and using the SU($d = 2$) lowering operator, identify the wave functions of all other states at that energy level having the same "spin" value T but different values of $T_3 = p$. Repeating the same procedure over and over again, starting each time from the remaining highest spin state, the whole content of states at energy level E_n can be exhausted that way. However, a more efficient approach would be to introduce into the partition and generating functions a "tagging" in terms of the SU($d = 2$) Casimir operator, thereby immediately associating their T and T_3 values to each of the physical states and their wave functions.

Clearly, the same considerations using dynamical symmetries in conjunction with the physical projector operator and coherent states, apply to the general classes of gauge invariant quantum mechanical models defined by (110), and beyond those, to the genuine gauge invariant field theories mentioned previously. Progress along these lines is left for future work.

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Appendix A

Consider the ordinary one dimensional harmonic oscillator, of mass m and angular frequency ω. Its quantum Hamiltonian $\hat{H}$,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2 = \hbar\omega(a^\dagger a + \frac{1}{2}) \quad , \tag{111}$$

is given in terms of the pairs of conjugate degrees of freedom, either the (position, momentum) operators $\hat{q}$ and $\hat{p}$, or the (annihilation, creation) operators a and $a^\dagger$, obeying the usual algebraic relations

$$\hat{q}^\dagger = \hat{q} \quad , \quad \hat{p}^\dagger = \hat{p} \quad , \quad [\hat{q}, \hat{p}] = i\hbar \quad , \quad [a, a^\dagger] = 1 \quad . \tag{112}$$

The relations between these different operators are

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left[\hat{q} + i\frac{\hat{p}}{m\omega}\right] \quad , \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left[\hat{q} - i\frac{\hat{p}}{m\omega}\right] \quad , \tag{113}$$

$$\hat{q} = \sqrt{\frac{\hbar}{2m\omega}}\left[a + a^\dagger\right] \quad , \quad \hat{p} = -i\sqrt{\frac{m\hbar\omega}{2}}\left[a - a^\dagger\right] \quad . \tag{114}$$

As is well known, the Fock space representation of the $(a, a^\dagger)$ algebra is spanned by a basis of orthonormalised states $|n>$ $(n = 0, 1, \ldots)$ which also diagonalise the quantum Hamiltonian $\hat{H}$,

$$|n> = \frac{1}{\sqrt{n!}}\,(a^\dagger)^n\,|0> \quad , \quad \hat{H}|n> = \hbar\omega(n + \frac{1}{2})|n> \quad , \quad n = 0, 1, 2, \ldots \quad , \tag{115}$$

$|0>$ being the normalised vacuum state annihilated by the operator a, while the states $|n>$ are thus such that $< n|m > = \delta_{nm}$.

By analogy with (113) and (114), and given real values q and p related to the phase space degrees of freedom of the system, let us introduce the complex quantities

$$z = \sqrt{\frac{m\omega}{2\hbar}}\left[q + i\frac{p}{m\omega}\right] \quad , \quad \bar{z} = \sqrt{\frac{m\omega}{2\hbar}}\left[q - i\frac{p}{m\omega}\right] \quad , \tag{116}$$

so that,

$$q = \sqrt{\frac{\hbar}{2m\omega}}\,[z + \bar{z}] \quad , \quad p = -i\sqrt{\frac{m\hbar\omega}{2}}\,[z - \bar{z}] \quad . \tag{117}$$

Associated to these quantities, one introduces the holomorphic and phase space coherent states, defined, respectively, by

$$|z> = e^{-\frac{1}{2}|z|^2}\,e^{za^\dagger}\,|0> \quad , \quad |p, q> = e^{-\frac{i}{\hbar}q\hat{p}}\,e^{\frac{i}{\hbar}p\hat{q}}\,|0> \quad . \tag{118}$$

Given the identity,

$$za^\dagger - \bar{z}a = -\frac{i}{\hbar}q\hat{p} + \frac{i}{\hbar}p\hat{q} \quad , \tag{119}$$

these two sets of coherent states are essentially equivalent up to a phase factor, as follows,

$$|z> = e^{\frac{i}{2\hbar}qp}\,|p, q> \quad , \tag{120}$$

provided of course the variables z, q and p are related as in (116) and (117).

The interest of these coherent states lies with the fact that they also generate the whole representation space of the quantum oscillator. Indeed, the identity operator possesses the following resolutions,

$$\mathbb{1}=\sum_{n=0}^{\infty}|n><n|=\int\frac{dzd\bar{z}}{\pi}\,|z><z|=\int_{(\infty)}\frac{dqdp}{2\pi\hbar}\,|p,q><p,q|\quad . \tag{121}$$

Consequently, even though coherent states provide an overcomplete basis of states, they allow for more straightforward calculations. Indeed, quantum wave functions often reduce to simple polynomials in the variables z or $\bar{z}$, rather than special functions for the ordinary position or momentum wave function representations. Thus, the basic matrix elements of direct use are simply

$$a^n|z>=z^n|z>\quad,\quad <n|z>=\frac{1}{\sqrt{n!}}\,z^n\,e^{-\frac{1}{2}|z|^2}\quad,\quad n=0,1,2,\ldots\quad, \tag{122}$$

while one also has

$$<z_1|z_2>=e^{-\frac{1}{2}|z_1|^2-\frac{1}{2}|z_2|^2+\bar{z}_1z_2}\quad,\quad <z|z>=1\quad, \tag{123}$$

or equivalently

$$<p_2,q_2|p_1,q_1>=e^{-\frac{m\omega}{4\hbar}\left[(q_2-q_1)^2+\left(\frac{p_2-p_1}{m\omega}\right)^2-\frac{2i}{m\omega}(p_2+p_1)(q_2-q_1)\right]}\quad,\quad <p,q|p,q>=1\quad . \tag{124}$$

These expressions should be contrasted, for example, to the position wave functions of the energy eigenstates $|n>$,

$$<q|n>=\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\,\frac{1}{\sqrt{2^n n!}}\,e^{-\frac{m\omega}{2\hbar}q^2}\,H_n\left(q\sqrt{\frac{m\omega}{\hbar}}\right)\quad,\quad n=0,1,2,\ldots\quad, \tag{125}$$

$H_n(x)$ being the usual Hermite polynomials, and $|q>$ the orthonormalised basis of position eigenstates such that $<q|q'>=\delta(q-q')$.

Clearly, given the different results above, it is possible to construct the position wave function representation $<q|\psi>$ of any state $|\psi>$ from its coherent state wave function $<z|\psi>$ which typically, up to the gaussian factor $e^{-\frac{1}{2}|z|^2}$, is simply some polynomial in $\bar{z}$ (see (122) for an example).

Consider now two such harmonic oscillators, labelled by the indices $i=1,2$, with identical mass m and angular frequency ω, or in other words, a two dimensional harmonic oscillator with circular symmetry. All the above discussion goes through in that case in an obvious way, since the corresponding representation space is simply the tensor product, over the index values $i=1$ and $i=2$, of the previous one. Nevertheless, in order to make explicit the global SO(2)=U(1) circular symmetry of the system, it proves efficient to work, rather than with the cartesian basis (q_1,q_2), with a helicity-like basis of coordinates $q_\pm=(q_1\mp iq_2)/\sqrt{2}$. Indeed, SO(2) rotations of the cartesian coordinates by an angle θ then correspond to U(1) phase transformations $e^{\pm i\theta}$ of the helicity coordinates.

More specifically, rather than working with the usual annihilation and creation operator algebra $(a_i,a_i^\dagger)$ $(i=1,2)$, let us introduce the quantities

$$a_\pm=\frac{1}{\sqrt{2}}\left[a_1\mp ia_2\right]\quad,\quad a_\pm^\dagger=\frac{1}{\sqrt{2}}\left[a_1^\dagger\pm ia_2^\dagger\right]\quad, \tag{126}$$

and likewise, given complex variables z_i $(i=1,2)$ related to the cartesian phase space coordinates (q_i,p_i) $(i=1,2)$ in the same manner as above, consider the combinations

$$z_\pm=\frac{1}{\sqrt{2}}\left[z_1\mp iz_2\right]\quad,\quad \bar{z}_\pm=\frac{1}{\sqrt{2}}\left[\bar{z}_1\pm i\bar{z}_2\right]\quad . \tag{127}$$

Quite obviously, the ensuing helicity basis quantum algebra is still given by two sets of commuting creation and annihilation operators,

$$[a_+, a_-^\dagger] = 0 = [a_-, a_+^\dagger] \quad , \quad [a_+, a_+^\dagger] = 1 = [a_-, a_-^\dagger] \quad , \tag{128}$$

while the following relations also hold

$$z_1 a_1^\dagger + z_2 a_2^\dagger = z_+ a_+^\dagger + z_- a_-^\dagger \quad , \quad |z_1|^2 + |z_2|^2 = |z_+|^2 + |z_-|^2 \quad . \tag{129}$$

The quantum Hamiltonian then reads,

$$\hat{H} = \hbar\omega \left[a_1^\dagger a_1 + a_2^\dagger a_2 + 1 \right] = \hbar\omega \left[a_+^\dagger a_+ + a_-^\dagger a_- + 1 \right] \quad , \tag{130}$$

as well as the generator of global SO(2)=U(1) symmetry transformations,

$$Q = -i \left[a_1^\dagger a_2 - a_2^\dagger a_1 \right] = a_+^\dagger a_+ - a_-^\dagger a_- \quad , \tag{131}$$

such that,

$$[Q, a_\pm] = \mp a_\pm \quad , \quad [Q, a_\pm^\dagger] = \pm a_\pm^\dagger \quad . \tag{132}$$

Correspondingly, the Hamiltonian is diagonalised by the helicity Fock space basis,

$$|n_+, n_- >= \frac{1}{\sqrt{n_+!\; n_-!}} \left(a_+^\dagger \right)^{n_+} \left(a_-^\dagger \right)^{n_-} |0> \quad , \quad n_+, n_- = 0, 1, 2, \ldots \quad , \tag{133}$$

$$\hat{H}|n_+, n_- >= E(n_+, n_-)|n_+, n_- > \quad , \quad E(n_+, n_-) = \hbar\omega(n_+ + n_- + 1)|n_+, n_- > \quad , \tag{134}$$

while the helicity (holomorphic) coherent states are defined by,

$$|z_+, z_- >= e^{-\frac{1}{2}\left[|z_+|^2 + |z_-|^2\right]} \, e^{z_+ a_+^\dagger} \, e^{z_- a_-^\dagger} \, |0> \quad . \tag{135}$$

In terms of these generating vectors, the resolution of the identity is expressed as,

$$\mathbb{1} = \sum_{n_+, n_- = 0}^{\infty} |n_+, n_- >< n_+, n_-| = \int \prod_{\pm} \frac{dz_\pm d\bar{z}_\pm}{\pi} \, |z_+, z_- >< z_+, z_-| \quad , \tag{136}$$

with the basic matrix elements for the change of basis given by,

$$< n_+, n_-|z_+, z_- >= \frac{1}{\sqrt{n_+! n_-!}} \, z_+^{n_+} \, z_-^{n_-} \, e^{-\frac{1}{2}\left[|z_+|^2 + |z_-|^2\right]} \quad . \tag{137}$$

It also proves interesting to consider the action of the SO(2)=U(1) rotation generator Q on the coherent states. A straightforward analysis using (136) and (137) finds,

$$e^{i\theta Q}|z_+, z_- >= |e^{i\theta} z_+, e^{-i\theta} z_- > \quad , \tag{138}$$

thus confirming the above claim as to the property of the helicity basis under SO(2)=U(1) transformations. In terms of the cartesian variables z_i ($i = 1, 2$), the phase transformations $z'_\pm = e^{\pm i\theta} z_\pm$ correspond to the SO(2) rotation,

$$\begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad , \tag{139}$$

or equivalently in matrix notation, to $z'_i = U_{ij}(\theta) z_j$ $(i=1,2)$ with the rotation matrix,

$$[U_{ij}(\theta)] = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad . \tag{140}$$

Consequently, the action of the rotation generator on the coherent states $|z_i>$ in cartesian coordinates is given by,

$$e^{i\theta Q}|z_i> = |U_{ij}(\theta) z_j> \quad . \tag{141}$$

As a matter of fact, the two dimensional system possesses a global symmetry larger than the SO(2)=U(1) rotational one considered so far. Indeed, even though the *classical* harmonic oscillator is invariant under the global SO(2)=U(1) transformations constructed above, *at the quantum level*, it does possess a *dynamical* global unitary U(2)=SU(2)× $\mathrm{U}(1)_N$ symmetry extending the initial SO(2) one. This U(2) symmetry is related to the possibility, *appearing only at the quantum level*, to perform arbitrary U(2) rotations among the creation (or the annihilation) operators $a_i^\dagger$ (or a_i) $(i=1,2)$, or $a_\pm^\dagger$ (or $a_\pm$), while the Hamiltonian $\hat{H}$ in (130) is invariant under these unitary transformations. The generator associated to the $\mathrm{U}(1)_N$ factor of the U(2) symmetry is the excitation level operator $\hat{N}$, given by

$$\hat{N} = a_1^\dagger a_1 + a_2^\dagger a_2 = a_+^\dagger a_+ + a_-^\dagger a_- \quad . \tag{142}$$

The expression of the remaining SU(2) generators in terms of the creation and annihilation operators is defined up to an SU(2) homeomorphism, namely up to an arbitrary unitary linear combination of the cartesian or helicity coordinates. Necessarily, the generator Q of the original global SO(2)=U(1) symmetry is then also a certain specific linear combination in the SU(2) algebra, thereby determining the SO(2) embedding in SU(2).

The preferred choice of embedding is such that Q coincides with the third generator T^3 of SU(2), and is best expressed in the helicity basis. Given the usual Pauli matrices τ^a $(a=1,2,3)$,

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \tag{143}$$

a basis of SU(2) generators is specified by[22]

$$T^a = \sum_{\eta,\eta'=+,-} a_{\eta'}^\dagger \left(\frac{\tau^a}{2}\right)_{\eta',\eta} a_\eta = a^\dagger \cdot \frac{\tau^a}{2} \cdot a \quad , \tag{144}$$

whose algebra is thus,

$$T^{a\dagger} = T^a \quad , \quad [T^a, T^b] = i\epsilon^{abc} T^c \quad , \quad a,b,c = 1,2,3 \quad . \tag{145}$$

In particular, the raising and lowering operators $T^\pm = T^1 \pm i T^2$ then reduce to

$$T^+ = a_+^\dagger a_- \quad , \quad T^- = a_-^\dagger a_+ \quad , \tag{146}$$

while T^3 then coincides indeed with the SO(2) rotation generator $Q/2$,

$$T^3 = \frac{1}{2}\left[a_+^\dagger a_+ - a_-^\dagger a_-\right] = \frac{1}{2} Q \quad . \tag{147}$$

[22] A similar definition is possible in terms of the cartesian creation and annihilation operators $a_i^\dagger$ and a_i $(i=1,2)$, in which case it is the generator T^2 which coincides with the SO(2)=U(1) charge $Q/2$.

The dynamical SU(2) symmetry in fact explains the degeneracies of the energy spectrum $E(n_+, n_-)$ in (134). For a fixed value of $n = n_+ + n_-$, corresponding to states $|n_+, n_- >$ sharing a common energy value $\hbar\omega(n+1)$, all these states span the n times totally symmetric SU(2) representation of dimension $(n+1)$, *i.e.* of "spin" $n/2$, while they are distinguished by their Q or $2T^3$ eigenvalue $(n_+ - n_-)$. Indeed, since the creation operators $a_+^\dagger$ and $a_-^\dagger$ commute and define the fundamental SU(2) representation, the states (133) at level $\hat{N} = n = n_+ + n_-$ all fall into the representation of "spin" $n/2$, and are related to one another by the action of the raising and lowering operators T^+ and T^-. Thus for example, the state $(a_-^\dagger)^n|0> /\sqrt{n!}$ defines the lowest weight state of the representation of "spin" $n/2$, of energy $E = \hbar\omega(n+1)$ and of "spin" projection $(-n/2)$, while all the other properly orthonormalised states of the same energy and of "spin" projection ranging from $(-(n-1)/2)$ up to $n/2$, are obtained by the repeated action—up to n times—of the simple positive root T^+.

Such dynamical symmetries are characteristic of spherical harmonic oscillators in any d dimensional Euclidean space. Indeed, beyond the classical SO(d) symmetry acting on the cartesian coordinates parametrising such systems, at the quantum level, U(d)=SU(d)$\times$U$(1)_N$ unitary transformations among the cartesian creation (or annihilation) operators $a_i^\dagger$ $(i = 1, 2, \cdots, d)$ define a dynamical symmetry commuting with the quantum Hamiltonian $\hat{H}$, which, up to a constant and a factor, coincides with the excitation level operator $\hat{N}$ generating the U$(1)_N$ symmetry,

$$\hat{H} = \hbar\omega \left[\sum_{i=1}^{d} a_i^\dagger a_i \, + \frac{1}{2}d\right] = \hbar\omega \left[\hat{N} + \frac{1}{2}d\right] \quad . \tag{148}$$

Given the algebra

$$[a_i, a_j^\dagger] = \delta_{ij} \quad , \quad i, j = 1, 2, \cdots, d \quad , \tag{149}$$

holomorphic coherent states are defined in the same way as in the one dimensional case, in terms of a collection of complex variables z_i which may also be related to cartesian phase space coordinates q_i and p_i $(i = 1, 2, \cdots, d)$,

$$|z_i >= e^{-\frac{1}{2}|z_i|^2} e^{z_i a_i^\dagger}|0> \quad . \tag{150}$$

In order to define the action of the dynamical SU(d) symmetry on these coherent states, since the creation (or annihilation) operators $a_i^\dagger$ transform in the fundamental SU(d) representation of dimension d, let us choose a set of $d \times d$ hermitian matrices T^a $(a = 1, 2, \cdots, d^2 - 1)$ generating the SU(d) transformations in the fundamental representation, and construct the following operators,

$$Q^a = a_i^\dagger \, T_{ij}^a \, a_j \quad , \quad Q^{a\,\dagger} = Q^a \quad . \tag{151}$$

Given the SU(d) algebra,

$$[T^a, T^b] = i f^{abc} T^c \quad , \tag{152}$$

as well as the Jacobi identity for the structure coefficients f^{abc}, it is possible to show that the quantities Q^a $(a = 1, 2, \cdots, d^2 - 1)$ do indeed generate the same SU(d) algebra, with the creation and annihilation operators transforming in the fundamental representation,

$$[Q^a, Q^b] = i f^{abc} Q^c \quad , \quad [Q^a, a_i^\dagger] = a_j^\dagger \, T_{ji}^a \quad , \quad [Q^a, a_i] = -T_{ij}^a a_j \quad . \tag{153}$$

Consider then finite SU(d) transformations acting on these operators,

$$e^{i\theta^a Q^a} \, a_i^\dagger \, e^{-i\theta^a Q^a} = a_j^\dagger \, U_{ji}(\theta^a) \quad , \tag{154}$$

where the matrices $U_{ij}(\theta^a)$ thus define the finite SU(d) group transformations in the fundamental representation,

$$U_{ij}(\theta^a) = \left(e^{i\theta^a T^a}\right)_{ij} \quad . \tag{155}$$

It is then straightforward to show that the coherent states as well do indeed transform in the same fundamental SU(d) representation, namely,

$$e^{i\theta^a Q^a}\, |z_i> = |U_{ij}(\theta^a) z_j> \quad , \tag{156}$$

a result which thus generalises that of (141) in the case of the global SO(2)=U(1) symmetry of the two dimensional spherical harmonic oscillator.

Finally, it should also be clear that the whole degeneracy of the quantum Hamiltonian $\hat{H}$ in (148) at a given energy level of excitation $\hat{N} = n$, is directly accounted for in terms of the n times totally symmetric SU(d) representation, of dimension $(d-1+n)!/\left((d-1)!\,n!\right)$, whose set of states is thus generated through the repeated application of the SU(d) simple positive roots on the lowest weight state in that representation.

Appendix B

Consider the following Euler angle parametrisation of SO(3) transformations in the defining vector representation,

$$U(\psi,\theta,\phi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \tag{157}$$

where the range of variation for the Euler angles is,

$$0 \le \psi \le 2\pi \quad , \quad 0 \le \theta \le \pi \quad , \quad 0 \le \phi \le 2\pi \quad . \tag{158}$$

Introducing the canonical basis of SO(3) generators $(T_i)_{jk} = -i\epsilon_{ijk}$, namely,

$$T_1 = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad , \quad T_2 = -i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad , \quad T_3 = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \tag{159}$$

the (left-)invariant Haar integration measure over SO(3) is then given by the determinant of the matrix defined by considering the Lie algebra valued differential $\left(-iU^{-1}(\psi,\theta,\phi)dU(\psi,\theta\phi)\right)$ expanded both in the Euler angle parametrisation and in the Lie algebra generators. After some algebra, one then finds that the normalised Haar measure over SO(3) is given by,

$$\int_{\mathrm{SO}(3)} d\mu = \frac{1}{8\pi^2} \int_0^{2\pi} d\psi \int_0^{\pi} d\theta\, \sin\theta \int_0^{2\pi} d\phi \quad . \tag{160}$$

Another result of use is the expression for the determinant $\det\Big[\mathbb{1} - x\,U(\psi,\theta,\phi)\Big]$, where x is some arbitrary parameter. Straightforward algebra leads to the expression,

$$\det\Big[\mathbb{1} - x\,U(\psi,\theta,\phi)\Big] = (1-x)\Big\{(1+x+x^2) + x\left[1 - (1+\cos\theta)(1+\cos(\psi+\phi))\right]\Big\} \quad . \tag{161}$$

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